PROCEEDING

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INTERNATIONAL CONFERENCE ON RESEARCH, IMPLEMENTATION AND EDUCATION OF MATHEMATICS AND SCIENCES 2014

ICRIEMS 2014

Yogyakarta, 18-20 May 2014

Global Trends and Issues on Mathematics and Sciences and the Education

PROCEEDING

Conference Proceedings

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Global Trends and Issues
on Mathematics and Science
and The Education

Faculty of Mathematics and Natural Sciences
Yogyakarta State University

ICRIEMS 2014 : Global Trends and Issues on Mathematics and Science and The Education

- Mathematics & Mathematics Education
- Physics & Physics Education
- Chemistry & Chemistry Education
- Biology & Biology Education
- Science Education

Preface

Bless upon God Almighty such that this proceeding on International Conference on Research, Implementation, and Education of Mathematics and Sciences (ICRIEMS) 2014 may be compiled according to the schedule provided by the organizing committee. All of the articles in this proceeding are obtained by selection process by the reviwer team and already ben presented in the Conference on 18 – 20 May 2014 in the Faculty of Mathematics and Natural Sciences, Yogyakarta State University. This proceeding consists of 344 parallel papers, and comprises 9 fields, that is mathematics, mathematics education, physics, physics education, chemistry, chemistry education, biology, biology education, and science education.

The theme of ICRIEMS 2014 is 'Global Trends and Issues of Mathematics and Science and the Education'. The main articles in this conference are given by five keynote speakers, which are Prof. Dean Zollman (Physics Department, Kansas State University), Prof. David F. Treagust (Center of Education, Curtin University), Prof. Dr. Amy Cutter-Mackenzie (School of Education, Southern Cross University, Australia), Prof. Tran Vui (Hue University, Vietnam), and Asst. Prof. Dr. Duangjai Nacapricha (Faculty of Science, Mahidol University). The conference is also supported by the LPTK (Lembaga Pendidikan Tenaga Kependidikan) Forum from Faculty of Mathematics and Sciences that consists of 12 universities all over Indonesia. Each member of the Forum contributed one invited speakers, such that there are an additional 10 invited speakers presenting in the forum. Besides the keynote and invited speakers, there are also 344 parallel articles that presented the latest research results in the field of mathematics and sciences, and the education. These parallel session speakers come from researchers from Indonesia and abroad, including Malaysia and Australia.

Hopefully, this proceeding may contribute in disseminating research results and studies in the field of Mathematics and Sciences and the Education such that they are accessible by many people and useful for the Nation Building.

Yogyakarta, June 2014

The Editor Team

Forewords from The Head of Committee

Assalamu'alaikum wa Rahmatullahi wa Barakatuh
May God bless upon us.

Your excellency The president of UNY Prof. Dr. Rochmat Wahab, M. Pd., M.A., ladies and gentlemen, good morning and welcome to State University Yogyakarta. This seminar entitled International Conference on Research, Implementation, and Education of Mathematics and Science (ICRIEMS): global trends and issues on mathematics and science and the education is organized by the Faculty of Mathematics and Science, State University of Yogyakarta working together with 12 members of the Association of the Faculty of Math and Sciences from Teacher Education Program (LPTK). This seminar is also dedicated to the golden aniversary of UNY; 1 among 90 academic activities dedicated to the aniversary.

Ladies and gentlemen, on behalf of the committee of this conference, I would like to express highest appreciation and gratitudes to the keynote speakers, including:

1. Prof. David F. Treagust (Center of Science Education Curtin University)
2. Prof. Dean Zollman (Physics Dept, Kansas University, US)
3. Dr. Amy Cutter-Mackenzie (School of Education, Southern Cross University, Australia)
4. Asst. Prof. Dr. Duangjai Nacapricha (Faculty of Science , Mahidol University)
5. Prof. Tran Vui (College of Education, Hue University, Hue City, Vietnam)

Secondly, I would like also to give sincere thanks and gratitudes to the speakers from 10 College of Educations, including:

1. Universitas Negeri Surabaya (UNESA): Prof. Dr. Muchlas Samani, and 33 speakers
2. Universitas Negeri Jakarta (UNJ): Prof. Dr. Gerardus Pola, and 7 speaker
3. Universitas Pendidikan Indonesia (UPI): Dr. Hary Firman, and
4. Universitas Negeri Malang (UM): Prof. Effendi, Ph.D
5. Universitas Negeri Padang (UNP): Prof. Tjeerd Plomp
6. Universitas Negeri Semarang (UNNES): Prof. Dr. Supriyadi Rustad

7. Universitas Pendidikan Singaraja (UNDIKSA): Prof. Dr. I Nengah Suparta, M.Si
8. Universitas Negeri Makasar (UNM): Oslan Junaidi, Ph.D
9. Universitas Negeri Gorontalo (UNG): Prof. Dr. Sarson Pomalto, M.Pd
10. Universitas Negeri Yogyakarta (UNY): Dr. Jaslin Ikhsan

Next, I also would like to thanks to our special guests and speakers from:

1. Universitas Pendidikan Sultan Indris (UPSI), Malaysia
2. University of Mahidol, Thailand
3. University of Malaysia in Trengganu

Next, I would like to thanks and welcome to 379 speakers from the entire Indonesia and all participants registered in this seminar.

Ladies and gentlemen, recently the number of research and publication on mathematics and science and the education is vulnarable. It is nescessary for us to organise, to share, and to publish the results of the research in this conference. I hope the conference will bear fruitful results and promote networking and future collaborations for all participants from diverse background of expertise, intitutions, and countries to promote science, mathematics, and the education.

Finally, I am delighted to thank the committee members who have been working very hard to ensure the succes of the conference.

Please enjoy the conference and enjoy Yogyakarta, the city of education, tourism, and culture. Thank you very much.

Assalamu'alaikum wa rahmatullahi wa barrakatuh

Dr. Slamet Suyanto, M. Ed.

Forewords from The Dean of Faculty of Mathematics and Natural Sciences, Yogyakarta State University

Assalamu'alaikum warahmatullahi wabarakatuh

May peace and God's blessings be upon us all.

On behalf of the Organizing Committee, first of all allow me to extend my warmest greeting and welcome to the International Conference on Research, Implementation, and Education of Mathematics and Sciences 2014, held in Yogyakarta State University, one of the qualified education universities in Indonesia.

To celebrate the 50th Commemoration of Yogyakarta State University, our faculty, in collaboration with Forum of MIPA LPTK, has the opportunity to conduct International Conference on Research, Implementation, and Education of Mathematics and Sciences 2014. This conference proudly presents five keynote speeches by five fabulous speakers: Prof. Dean Zollman, Prof. David F. Treagust, Prof. Dr. Amy Cutter-Mackenzie, Prof. Tran Vui, and Asst. Prof. Dr. Duangjai Nacapricha, around 380 parallel speakers with 344 orally presented articles.

Distinguished guest, ladies and gentlemen,

The independence of a country is impossible to gain if the education does not become the priority and it is not supported with the development of technology. We all know that the technology development could be achieved if it is supported by the improvement of firm fundamental knowledge. The empowerment of fundamental knowledge could not be separated from research which is related to the development of technology and the learning process in school and universities.

This conference is aimed to pull together researchers, educators, policy makers, and practitioners to share their critical thinking and research outcomes. Therefore, we are able to understand and examine the development of fundamental principle, knowledge, and technology. By perceiving the matters and condition in research and education field of mathematics and sciences, we could take a part in conducting qualified education to reach out the real independence of our nation.

Distinguished guest, ladies, and gentlemen

This conference will be far from success and we could not accomplish what we do without the support from various parties. So let me extend my deepest gratitude and highest appreciation to all committee members. I would also like to thank each of participants for attending our conference and bringing your expertise to our gathering. Should you find any inconveniences and shortcomings, please accept my sincere apologies.

To conclude, let me wish you fruitful discussion and a very pleasant stay in Yogyakarta.

Wa'alaikumsalam warahmatullahi wabarakatuh

Dr. Hartono

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SYSTEMS OF INTERVAL MIN-PLUS LINEAR EQUATIONS AND ITS APPLICATION ON SHORTEST PATH PROBLEM WITH INTERVAL TRAVEL TIMES

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Abstract

The travel times in a network are seldom precisely known, and then could be represented into the interval of real number, that is called interval travel times. This paper discusses the solution of the iterative systems of interval min-plus linear equations its application on shortest path problem with interval travel times. The finding shows that the iterative systems of interval min-plus linear equations, with coefficient matrix is semi-definite, has a maximum interval solution. Moreover, if coefficient matrix is definite, then the interval solution is unique. The networks with interval travel time can be represented as a matrix over interval min-plus algebra. The networks dynamics can be represented as an iterative system of interval min-plus linear equations. From the solution of the system, can be deter-mined interval earliest starting times for each point can be traversed. Furthermore, we can determine the interval fastest time to traverse the network. Finally, we can determine the shortest path interval with interval travel times by determining the shortest path with crisp travel times.

Key words: Min-Plus Algebra, Linear System, Shortest Path, Interval.

INTRODUCTION

Let $\mathbf{R}_\varepsilon := \mathbf{R} \cup \{\varepsilon\}$ with $\mathbf{R}$ the set of all real numbers and $\varepsilon : = \infty$. In $\mathbf{R}_\varepsilon$ defined two operations : $\forall a,b \in \mathbf{R}_\varepsilon$, $a \oplus b := \min(a, b)$ and $a \otimes b : = a + b$. We can show that $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a commutative idempotent semiring with neutral element $\varepsilon = \infty$ and unity element $e = 0$. Moreover, $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a semifield, that is $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a commutative semiring, where for every $a \in \mathbf{R}$ there exist $-a$ such that $a \otimes (-a) = 0$. Thus, $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is a *min-plus algebra*, and is written as $\mathbf{R}_{\min}$. One can define $x^{\otimes^0} := 0$, $x^{\otimes^k} := x \otimes x^{\otimes^{k-1}}$, $\varepsilon^{\otimes^0} : = 0$ and $\varepsilon^{\otimes^k} : = \varepsilon$, for k = 1, 2, ... The operations $\oplus$ and $\otimes$ in $\mathbf{R}_{\min}$ can be extend to the matrices operations in $\mathbf{R}_{\min}^{m\times n}$, with $\mathbf{R}_{\min}^{m\times n} : = \{A = (A_{ij}) \mid A_{ij} \in \mathbf{R}_{\min}$, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n\}$, the set of all matrices over max-plus algebra. Specifically, for $A, B \in \mathbf{R}_{\min}^{n\times n}$ we define $(A \oplus B)_{ij} = A_{ij} \oplus B_{ij}$ and $(A \otimes B)_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}$. We also define matrix $E \in \mathbf{R}_{\min}^{n\times n}$, $(E)_{ij} : = \begin{cases} 0 \ , \text{ if } i = j \\ \varepsilon \ , \text{if } i \neq j \end{cases}$ and $\varepsilon \in \mathbf{R}_{\min}^{m\times n}$, $(\varepsilon)_{ij}$ $:= \varepsilon$ for every i and j . For any matrices $A \in \mathbf{R}_{\min}^{n\times n}$, one can define $A^{\otimes^0} = E_n$ and $A^{\otimes^k} = A \otimes A^{\otimes^{k-1}}$ for k = 1, 2, ... For any weighted, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ we can define a matrix $A \in$

$\mathbf{R}_{\min}^{n\times n}$, $A_{ij} = \begin{cases} w(j,i), \text{ if } (j,i)\in \mathcal{A} \\ \varepsilon, \quad \text{ if } (j,i)\notin \mathcal{A}. \end{cases}$, called the *weight-matrix* of graph $\mathcal{G}$.

A matrix $A \in \mathbf{R}_{\min}^{n\times n}$ is said to be *semi-definite* if all of circuit in $\mathcal{G}(A)$ have nonnegative weight, and it is said *definite* if all of circuit in $\mathcal{G}(A)$ have positive weight. We can show that if any matrices A is semi-definite, then $\forall p \geq n$, $A^{\otimes p} \preceq_{\mathrm{m}} E \oplus A \oplus \ldots \oplus A^{\otimes n-1}$. So, we can define $A^* := E \oplus A \oplus \ldots \oplus A^{\otimes n} \oplus A^{\otimes n+1} \oplus \ldots$. Define $\mathbf{R}_{\min}^{n} := \{ \boldsymbol{x} = [x_1, x_2, \ldots , x_n]^{\mathrm{T}} \mid x_i \in \mathbf{R}_{\min}, i = 1, 2, \ldots , n\}$. Notice that we can be seen $\mathbf{R}_{\min}^{n}$ as $\mathbf{R}_{\min}^{n\times 1}$. The elements of $\mathbf{R}_{\max}^{n}$ is called *vector* over $\mathbf{R}_{\min}$. In general, min-plus algebra is analogous to max-plus algebra. Further details about max-plus algebra, matrix and graph can be found in Baccelli *et.al* (2001) and Rudhito (2003).

The existence and uniqueness of the solution of the iterative system of min-plus linear equation and its application to determine the shortest path in the with crisp (real) travel times had been discussed in Rudhito (2013). The followings are some result in brief. Let $A \in \mathbf{R}_{\min}^{n\times n}$ and $\boldsymbol{b} \in \mathbf{R}_{\min}^{n\times 1}$. If A is semi-definite, then $\boldsymbol{x}^* = A^* \otimes \boldsymbol{b}$ is a solution of system $\boldsymbol{x} = A \otimes \boldsymbol{x} \oplus \boldsymbol{b}$. Moreover, if A is definite, then the system has a unique solution. A *one-way path network S* with crisp activity times, is a directed, strongly connected, acyclic, crisp weighted graph $S = (\mathcal{V}, \mathcal{A})$, with $V = \{1, 2, , \ldots , n\}$ suct that if $(i, j) \in \mathcal{A}$, then $i < j$. In this network, point represent *crosspathway*, arc expresses a *pathway*, while the weight of the arc represent *travel time*, so that the weights in the network is always positive. Let x_i^e is *earliest starting times* for point i can be traversed and $\boldsymbol{x}^{\boldsymbol{e}} = [x_1^e, x_2^e, \ldots , x_n^e]^{\mathrm{T}}$. For the network with crisp travel times, with n nodes and A the weight matrix of graph of the networks, then

$$\boldsymbol{x}^{\boldsymbol{e}} = (E \oplus A \oplus \ldots \oplus A^{\otimes n-1}) \otimes \boldsymbol{b}^{\boldsymbol{e}} = A^* \otimes \boldsymbol{b}^{\boldsymbol{e}}$$

with $\boldsymbol{b}^{\boldsymbol{e}} = [0, \varepsilon, \ldots , \varepsilon]^{\mathrm{T}}$. Furthermore, x_n^e is the *fastest times to traverse* the network. Let x_i^l is be *latest times left* point i and $\boldsymbol{x}^l = [x_1^l, x_2^l \; \ldots. , x_n^l]$. For the network above, vector

$$\boldsymbol{x}^l = - ((A^{\mathrm{T}})^* \otimes \boldsymbol{b}^l)$$

with $\boldsymbol{b}^{\boldsymbol{l}} = [\varepsilon, \varepsilon, \ldots , - x_n^e]^{\mathrm{T}}$. Define, a pathway $(i, j) \in \mathcal{A}$ in the one-way path network $\mathcal{S}$ is called *shortest pathway* if $x_i^e = x_i^l$ dan $x_j^e = x_j^l$. Define, A path $p \in P$ in the one-way path network $\mathcal{S}$ is called *shortest path* if all pathways belonging to p are shortest pathway. From this definition, we can show that a path $p \in P$ is a shortest path if and only if p has minimum weight, that is equal to x_n^e. Also, a pathway is a shortest pathway if and only if it belonging to a shortest path.

DISCUSSION

We discusses the solution of the iterative systems of interval min-plus linear equations its application on shortest path problem with interval travel times. The discussion begins by reviewing some basic concepts of interval min-plus algebra and matrices over interval min-plus algebra. Definition and concepts in the min-plus algebra analogous to the concepts in the max-plus algebra which can be seen in Rudhito (2011).

The (closed) interval x in $\mathbf{R}_{\min}$ is a subset of $\mathbf{R}_{\min}$ of the form

$$\mathrm{x} = [\underline{\mathrm{x}} , \overline{\mathrm{x}}] = \{x \in \mathbf{R}_{\min} \mid \underline{\mathrm{x}} \preceq_{\mathrm{m}} x \preceq_{\mathrm{m}} \overline{\mathrm{x}} \}.$$

The interval x in $\mathbf{R}_{\min}$ is called *min-plus interval*, which is in short is called *interval*. Define

$$\mathbf{I}(\mathbf{R})_\varepsilon := \{\, x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbf{R},\ \varepsilon \prec_m \underline{x} \preceq_m \overline{x} \,\} \cup \{\varepsilon\}, \text{ where } \varepsilon := [\varepsilon, \varepsilon].$$

In the $\mathbf{I}(\mathbf{R})_\varepsilon$, define operation $\overline{\oplus}$ and $\overline{\otimes}$ as

$$x \,\overline{\oplus}\, y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}] \text{ and } x \,\overline{\otimes}\, y = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}],\ \forall\, x, y \in \mathbf{I}(\mathbf{R})_\varepsilon.$$

Since $(\mathbf{R}_\varepsilon, \oplus, \otimes)$ is an idempotent semiring and it has no zero divisors, with neutral element ε, we can show that $\mathbf{I}(\mathbf{R})_\varepsilon$ is closed with respect to the operation $\overline{\oplus}$ and $\overline{\otimes}$. Moreover, $(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is a comutative idempotent semiring with neutral element $\varepsilon = [\varepsilon, \varepsilon]$ and unity element $0 = [0, 0]$. This comutative idempotent semiring $(\mathbf{I}(\mathbf{R})_\varepsilon, \overline{\oplus}, \overline{\otimes})$ is called *interval min-plus algebra* which is written as $\mathbf{I}(\mathbf{R})_{\min}$.

Define $\mathbf{I}(\mathbf{R})_{\min}^{m\times n} := \{A = (A_{ij}) \mid A_{ij} \in \mathbf{I}(\mathbf{R})_{\min}$, for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n\}$. The element of $\mathbf{I}(\mathbf{R})_{\min}^{m\times n}$ are called *matrices over interval min-plus algebra*. Furthermore, this matrices are called *interval matrices*. The operations $\overline{\oplus}$ and $\overline{\otimes}$ in $\mathbf{I}(\mathbf{R})_{\min}$ can be extended to the matrices operations of in $\mathbf{I}(\mathbf{R})_{\max}^{m\times n}$. Specifically, for A, B $\in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$ and $\alpha \in \mathbf{I}(\mathbf{R})_{\min}$ we define

$$(\alpha \,\overline{\otimes}\, A)_{ij} = \alpha \,\overline{\otimes}\, A_{ij},\ (A \,\overline{\oplus}\, B)_{ij} = A_{ij} \,\overline{\oplus}\, B_{ij} \text{ and } (A \,\overline{\otimes}\, B)_{ij} = \overline{\bigoplus_{k=1}^{n}} A_{ik} \,\overline{\otimes}\, B_{kj}.$$

Matrices A, B $\in \mathbf{I}(\mathbf{R})_{\min}^{m\times n}$ are *equal* if $A_{ij} = B_{ij}$, that is if $\underline{A_{ij}} = \underline{B_{ij}}$ and $\overline{A_{ij}} = \overline{B_{ij}}$ for every i and j. We can show that $(\mathbf{I}(\mathbf{R})_{\min}^{n\times n}, \overline{\oplus}, \overline{\otimes})$ is a idempotent semiring with neutral element is matrix ε, with $(\varepsilon)_{ij} := \varepsilon$ for every i and j, and unity element is matrix E, with $(E)_{ij} := \begin{cases} 0, & \text{if } i = j \\ \varepsilon, & \text{if } i \neq j \end{cases}$. We can also show that $\mathbf{I}(\mathbf{R})_{\min}^{m\times n}$ is a semi-module over $\mathbf{I}(\mathbf{R})_{\min}$.

For any matrix A $\in \mathbf{I}(\mathbf{R})_{\min}^{m\times n}$, define the matrices $\underline{A} = (\underline{A_{ij}}) \in \mathbf{R}_{\min}^{m\times n}$ and $\overline{A} = (\overline{A_{ij}}) \in \mathbf{R}_{\min}^{m\times n}$, which is called *lower bound matrices* and *upper bound matrices of* A, respectively. Define *matrices interval of* A, that is

$$[\underline{A}, \overline{A}] = \{\, A \in \mathbf{R}_{\min}^{m\times n} \mid \underline{A} \preceq_m A \preceq_m \overline{A} \,\} \text{ and } \mathbf{I}(\mathbf{R}_{\min}^{m\times n})^* = \{\, [\underline{A}, \overline{A}] \mid A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n} \,\}.$$

Specifically, for $[\underline{A}, \overline{A}], [\underline{B}, \overline{B}] \in \mathbf{I}(\mathbf{R}_{\min}^{m\times n})^*$ and $\alpha \in \mathbf{I}(\mathbf{R})_{\min}$ we define

$$\alpha \,\overline{\otimes}\, [\underline{A}, \overline{A}] = [\underline{\alpha} \otimes \underline{A}, \overline{\alpha} \otimes \overline{A}],\ [\underline{A}, \overline{A}] \,\overline{\oplus}\, [\underline{B}, \overline{B}] = [\underline{A} \oplus \underline{B}, \overline{A} \oplus \overline{B}]$$

$$\text{and } [\underline{A}, \overline{A}] \,\overline{\otimes}\, [\underline{B}, \overline{B}] = [\underline{A} \otimes \underline{B}, \overline{A} \otimes \overline{B}].$$

The matrices interval $[\underline{A}, \overline{A}]$ and $[\underline{B}, \overline{B}] \in \mathbf{I}(\mathbf{R}_{\min}^{m\times n})^*$ are *equal* if $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$. We can show that $(\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*, \overline{\oplus}, \overline{\otimes})$ is an idempotent semiring with neutral element matrix interval $[\varepsilon, \varepsilon]$ and the unity element is matrix interval [E, E]. We can also show that $\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$ is a semimodule over $\mathbf{I}(\mathbf{R})_{\min}$.

The semiring $(\mathbf{I}(\mathbf{R})_{\min}^{n\times n}, \overline{\oplus}, \overline{\otimes})$ is isomorfic with semiring $(\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*, \overline{\oplus}, \overline{\otimes})$. We can define a mapping f, where $f(A) = [\underline{A}, \overline{A}]$, $\forall A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$. Also, the semimodule $\mathbf{I}(\mathbf{R})_{\min}^{n\times n}$ is isomorfic with semimodule $\mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$. So, for every matrices interval A $\in \mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$ we can determine matrices interval $[\underline{A}, \overline{A}] \in \mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$. Conversely, for every $[\underline{A}, \overline{A}] \in \mathbf{I}(\mathbf{R}_{\min}^{n\times n})^*$, then

$\underline{A}, \overline{A} \in \mathbf{R}_{\min}^{n\times n}$, such that $[\underline{A}_{ij}, \overline{A}_{ij}] \in \mathbf{I}(\mathbf{R})_{\min}$, $\forall i$ and j. The matrix interval $[\underline{A}, \overline{A}]$ is called *matrix interval associated with the interval matrix* A and which is written $A \approx [\underline{A}, \overline{A}]$. So we have $\alpha \overline{\otimes} A \approx [\underline{\alpha} \otimes \underline{A}, \overline{\alpha} \otimes \overline{A}]$, $A \overline{\oplus} B \approx [\underline{A} \oplus \underline{B}, \overline{A} \oplus \overline{B}]$ and $A \overline{\otimes} B \approx [\underline{A} \otimes \underline{B}, \overline{A} \otimes \overline{B}]$.

We define for any interval matrices $A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$, where $A \approx [\underline{A}, \overline{A}]$, is said to be *semi-definite (definite)* if every matrices $A \in [\underline{A}, \overline{A}]$ is semi-definite (definite). We can show that interval matrices $A \in \mathbf{I}(\mathbf{R})_{\max}^{n\times n}$, where $A \approx [\underline{A}, \overline{A}]$ is semi-definite (definite) if and only if $\overline{A} \in \mathbf{R}_{\max}^{n\times n}$ semi-definite (definite).

Define $\mathbf{I}(\mathbf{R})_{\min}^{n} := \{ \mathbf{x} = [x_1, x_2, \ldots, x_n]^T \mid x_i \in \mathbf{I}(\mathbf{R})_{\min}, i = 1, 2, \ldots, n \}$. The set $\mathbf{I}(\mathbf{R})_{\min}^{n}$ can be seen as set $\mathbf{I}(\mathbf{R})_{\min}^{n\times 1}$. The Elements of $\mathbf{I}(\mathbf{R})_{\min}^{n}$ is called *interval vector over* $\mathbf{I}(\mathbf{R})_{\min}$. The interval vector $\mathbf{x}$ associated with *vector interval* $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$, that is $\mathbf{x} \approx [\underline{\mathbf{x}}, \overline{\mathbf{x}}]$.

Definition 1. *Let* $A \in \mathbf{I}(\mathbf{R})_{\min}^{n\times n}$ *and* $\mathbf{b} \in \mathbf{I}(\mathbf{R})_{\min}^{n}$. *A interval vector* $\mathbf{x}^* \in \mathbf{I}(\mathbf{R})_{\min}^{n}$ *is called* ***interval solution*** *of iterative system of interval min-plus linear equations* $\mathbf{x} = A \overline{\otimes} \mathbf{x} \overline{\oplus} \mathbf{b}$ *if* $\mathbf{x}^*$ *satisfy the system.*

Theorem 1. *Let* $A \in \mathbf{I}(\mathbf{R})_{\max}^{n\times n}$ *and* $\mathbf{b} \in \mathbf{I}(\mathbf{R})_{\min}^{n\times 1}$. *If* A *is semi-definite, then interval vector* $\mathbf{x}^* \approx [\underline{A}^* \otimes \underline{\mathbf{b}}, \overline{A}^* \otimes \overline{\mathbf{b}}]$, *is an interval solution of system* $\mathbf{x} = A \overline{\otimes} \mathbf{x} \overline{\oplus} \mathbf{b}$. *Moreover, if* A *is definite, then interval solution is unique*.

Proof. Proof is analogous to the case of max-plus algebra as seen in the Rudhito (2011)

Next will be discussed the *earliest starting times interval* for point *i* can be traversed. The discussion is analogous to the case of (crisp) travel time (Rudhito, 2013), using the interval min-plus algebra approach.

Let $ES_i = x_i^e$ is *earliest starting times interval* for point *i* can be traversed, with $x_i^e = [\underline{x}_i^e, \overline{x}_i^e]$.

$$A_{ij} = \begin{cases} \text{interval travel time from point } j \text{ to point } i & \text{if } (j,i) \in \mathcal{A} \\ \varepsilon (= [+\infty, +\infty]) & \text{if } (j,i) \notin \mathcal{A} \end{cases}.$$

We assume that $x_i^e = 0 = [0, 0]$ and with interval min-plus algebra notation we have

$$x_i^e = \begin{cases} 0 & \text{if } i = 1 \\ \bigoplus_{1 \le j \le n} (A_{ij} \overline{\otimes} x_j^e) & \text{if } i > 1 \end{cases}. \tag{1}$$

Let A is the interval weight matrix of the interval-valued weighted graph of the networks, $\mathbf{x}^e = [x_1^e, x_2^e, \ldots, x_n^e]^T$ dan $\mathbf{b}^e = [0, \varepsilon, \ldots, \varepsilon]^T$, then equation (1) can be written in an iterative system of interval max-plus linear equations

$$\mathbf{x}^e = A \overline{\otimes} \mathbf{x}^e \overline{\otimes} \mathbf{b}^e \tag{2}$$

Since the project networks is acyclic directed graph, then there are no circuit, so according to the result in Rudhito(2011), A is definite. And then according to Theorem 1,

$$\mathbf{x}^e = A^* \overline{\otimes} \mathbf{b}^e \approx [\underline{A}^* \otimes \underline{\mathbf{b}}^e, \overline{A}^* \otimes \overline{\mathbf{b}}^e]$$

$$= [(\underline{E} \oplus \underline{A} \oplus \ldots \underline{A}^{\otimes n-1}) \otimes \underline{\mathbf{b}}^{e}, (\overline{E} \oplus \overline{A} \oplus \ldots \oplus \overline{A}^{\otimes n-1}) \otimes \overline{\mathbf{b}}^{e}]$$

is a unique solution of the system (2), that is the vector of earliest starting times interval for point *i* can be traversed.

Notice that x_n^e is the *fastest times interval to traverse* the network. We summarize the description above in the Theorem 2.

Teorema 2. *Given a one-way path network network with interval travel times, with n node and* A *is the weight matrix of the interval-valued weighted graph of networks. The interval vector of earliest starting times interval for point i can be traversed is given by*

$$\mathbf{x}^{e} \approx [(\underline{E} \oplus \underline{A} \oplus \ldots \underline{A}^{\otimes n-1}) \otimes \underline{\mathbf{b}}^{e}, (\overline{E} \oplus \overline{A} \oplus \ldots \oplus \overline{A}^{\otimes n-1}) \otimes \overline{\mathbf{b}}^{e}]$$

with $\mathbf{b}^{e} = [0, \varepsilon, \ldots, \varepsilon]^{T}$. *Furthermore,* x_n^e *is the fastest times interval to traverse the network.*

Bukti: (see description above) . ■

Example 1 *Consider the project network in Figure 1.*

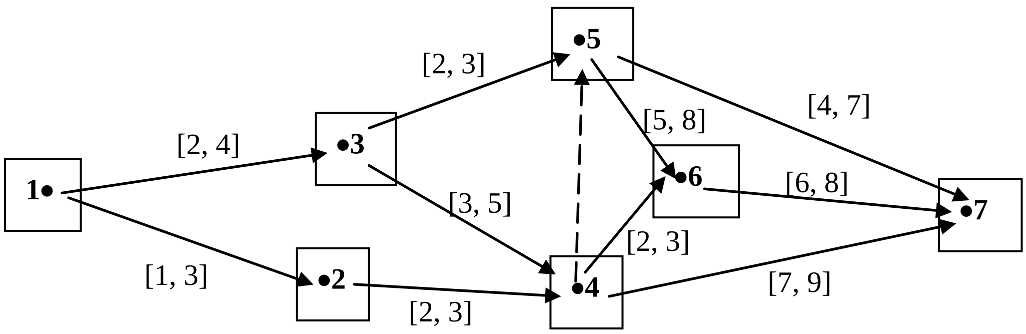

Figure 1. A one-way path network network with interval travel times

We have

$$A = \begin{bmatrix} \varepsilon, & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ [1,3] & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ [2,4] & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & [2,3] & [3,5] & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & [2,3] & [0,0] & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & [2,3] & [4,7] & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & [7,9] & [5,8] & [6,8] & \varepsilon \end{bmatrix}.$$

Using MATLAB computer program, we have

$$\underline{A}^* = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & 2 & 3 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 3 & 2 & 2 & 0 & 0 & \varepsilon & \varepsilon \\ 5 & 4 & 5 & 2 & 4 & 0 & \varepsilon \\ 8 & 7 & 7 & 5 & 5 & 6 & 0 \end{bmatrix}, \ \overline{A}^* = \begin{bmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 4 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 6 & 3 & 5 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 6 & 3 & 3 & 0 & 0 & \varepsilon & \varepsilon \\ 9 & 6 & 8 & 3 & 7 & 0 & \varepsilon \\ 14 & 11 & 11 & 8 & 8 & 8 & 0 \end{bmatrix},$$

$\underline{\mathbf{x}}^e = [0, 1, 2, 3, 3, 5, 8]^T$ dan $\overline{\mathbf{x}}^e = [0, 3, 4, 6, 6, 9, 14]^T$.

So the vector of earliest starting times interval for point i can be traversed is

$\mathbf{x}^e$ = [[0, 0], [1, 3], [2, 4], [3, 6], [3, 6], [5, 9], [8, 14]]^T and the fastest times interval to traverse the network x_n^e = [16, 25].

Next given shortest path interval definition and theorem that gives way determination. Definitions and results is a modification of the definition of critical path-interval and theorem to determine the critical path method-interval, as discussed in Chanas and Zielinski (2001) and Rudhito (2011).We also give some examples for illustration.

Definition 2. *A path $p \in P$ is called an* ***interval-shortest path*** *in* ***S*** *if there exist a set of travel times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that p is shortest path, after replacing the interval travel times* A_{ij} *with the travel time A_{ij}.*

Definisi 3. *A pathway $(k, l) \in \mathcal{A}$ is called an* ***interval-shortest pathway*** *in* ***S*** *if there exist a set of travel times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that (k, l) is shortest pathway, after replacing the interval travel times* A_{ij} *with the travel time A_{ij}.*

The following theorem is given which relates the interval-shortest path and interval-shortest pathway.

Teorema 3. *If path $p \in P$ is an interval-shortest path, then all pathways in the p are interval-shortest pathway.*

Proof : Let path $p \in P$ is an interval shortest path, then according to Definition 2, there exist a set of times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that p is shortest path, after replacing the interval travel times A_{ij} with the travel time A_{ij}. Next, according to the definition of shortest path above, all pathways in p are shortest pathways for a set of travel times $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$. Thus according to Definiton 3, all pathways in p are interval-shorstest pathways. ■

The following theorem is given a necessary and sufficient condition a path is an interval-shortest path.

Teorema 4. *A path $p \in P$ is an* ***interval-shortest path*** *in* ***S*** *if and only if p is a shortest path, with interval travel times* $\mathrm{A}_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, *have been replace with travel times A_{ij} which is determined by the following formula*

$$A_{ij} = \begin{cases} \overline{A}_{ij} \text{ jika } (i,\ j) \notin p \\ \underline{A}_{ij} \text{ jika } (i,\ j) \in p \end{cases}. \qquad (3)$$

Bukti : $\Rightarrow$: Let p is an interval-shortest path, then according to Definition 2, there exist a set of travel times A_{ij} , $A_{ij} \in [\underline{A}_{ij}, \overline{A}_{ij}]$, $(i, j) \in \mathcal{A}$, such that p is shortest pathway, after replacing the interval travel times A_{ij} with travel times A_{ij} , $(i, j) \in \mathcal{A}$. If the travel times for all pathway is located at p is reduced from A_{ij} to $\underline{A}_{ij}$ and for all pathway is not located p is increased from A_{ij} to $\overline{A}_{ij}$, then p is a path with minimum weight in $\boldsymbol{S}$ for new travel time formation. Thus path p is a shortest path.
$\Leftarrow$: Since path p a shortest path with a set of travel times $A_{ij} \in [\underline{\mathrm{A}}_{ij}, \overline{\mathrm{A}}_{ij}]$, which is determined by the formula (9), then according to Definition 2, path p is an interval-shortest path. ■

Example 2. We consider the network in Example 1. We will determine all interval-shortest path in this network. For path 1→3→5→7, by applying formula (9), we have weight

$$\begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & 5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 2 & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 3 & 8 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 9 & 4 & 8 & \varepsilon \end{bmatrix}.$$

Using MATLAB computer program, we have a shortest path 1→3→5→7 with minimum weight of path is 8. Thus 1→3→5→7 is an interval-shortest path. The results of the calculations for all possible path in the network are given in Table 1 below.

Tabel 1 Calculation results of all path

No	**Path *p***	**Weight Interval *p***	**Shortest-path p^*** (with formula (9))	**Weight of p^***	**Conclusion**
1	1→3→5→7	[8,14]	1→3→5→7 ,	8	Interval-shortest
2	1→3→5→6→7	[15, 23]	1→3→5→7	11	Not interval-shortest
3	1→3→4→5→7	[9, 16]	1→3→4→5→7 1→3→5→7	9	Interval-shortest
4	1→3→4→5→6→7	[16, 25]	1→3→4→5→7 1→3→5→7	12	Not interval-shortest
5	1→3→4→6→7	[13, 20]	1→3→4→5→7 1→3→5→7	12	Not interval-shortest
6	1→3→4→7	[12, 18]	1→3→4→5→7 1→3→4→7 1→3→5→7	12	Interval-shortest

7	1→2→4→5→7	[7, 13]	1→2→4→5→7	7	Interval-shortest
8	1→2→4→5→6→7	[14, 22]	1→2→4→5→7	10	Not interval-shortest
9	1→2→4→6→7	[11, 17]	1→2→4→5→7	10	Not interval-shortest
10	1→2→4→7	[10, 15]	1→2→4→5→7 1→2→4→7	10	Interval-shortest

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