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Systems of Max-Plus Linear Equations with More Variables than Equations

Regina Wahyudyah Sonata Ayu[1] and Marcellinus Andy Rudhito[1]

[1]Department of Mathematics Education, Sanata Dharma University, Yogyakarta Indonesia

E-mail: reginawahyudyahayu@gmail.com, rudhito@usd.ac.id

Abstract. This paper discusses the solution of systems of max-plus linear equations with more variables than equations through the reduced discrepancy matrix of the system. Let the entries of each column of the coefficient matrix are not all equal to infinite. We show that if there is a zero-row in reduced discrepancy matrix of the system, then the systems has no solution. Furthermore, if there is no zero-row in reduced discrepancy matrix of the system, then there are infinitely many solutions of the system.

Key words: Max-Plus algebra, system of linear equations, reduced discrepancy matrix.

1. Introduction

As in conventional algebra, we can also find system of linear equations in max-plus algebra. System of max-plus linear equations can also be represented by a matrix equation that is $A \otimes \boldsymbol{x} = \boldsymbol{b}$.

The solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in max-plus algebra through the reduced "*discrepancy*"matrix has been discussed in [1] and [4]. However, they just concern about the existence and the uniqueness of the solution to $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in general. They haven't concerned about the solution of the system of max-plus linear equations with more variables than equations in spesific yet. Therefore, in this article, we will discuss about the solution of the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$ with more variables than equations.

First, we will review some basic concepts of max-plus algebra, matrices over max-plus algebra and the solution of the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$. Futher details can be found in [2] and [4].

Let $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$ where $\mathbb{R}$ is a set of all real numbers and $\varepsilon := -\infty$. Defined two operations $\oplus$ dan $\otimes$ on $\mathbb{R}_\varepsilon$ such that

$$a \oplus b := max(a, b) \quad \text{dan} \quad a \otimes b := a + b \qquad , \forall a, b \in \mathbb{R}_\varepsilon.$$

$\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$ is a commutative idempotent semiring. Furthermore, $\mathbb{R}_{\max}$ is a semifield. Then, $\mathbb{R}_{\max}$ is called as max-plus algebra. The relation " $\leq_{max}$ " on $\mathbb{R}_{max}$ defined by $a \leq_{max} b \Leftrightarrow a \oplus b = b$ is a partial order on $\mathbb{R}_{max}$.

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E-mail: reginawahyudyahayu@gmail.com, rudhito@usd.ac.id

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1. Introduction

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The solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in max-plus algebra through the reduced "*discrepancy*"matrix has been discussed in [1] and [4]. However, they just concern about the existence and the uniqueness of the solution to $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in general. They haven't concerned about the solution of the system of max-plus linear equations with more variables than equations in spesific yet. Therefore, in this article, we will discuss about the solution of the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$ with more variables than equations.

First, we will review some basic concepts of max-plus algebra, matrices over max-plus algebra and the solution of the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$. Futher details can be found in [2] and [4].

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The operations ⊕ dan ⊗ on $\mathbb{R}_{max}$ can be extended to set $\mathbb{R}_{\max}^{m\times n}$ where $\mathbb{R}_{\max}^{m\times n} = \{A = a_{ij} \mid a_{ij} \in \mathbb{R}_{max}, \text{for } i = 1,2, \ldots, m \text{ and } j = 1,2, \ldots, n\}$. Let $A, B \in \mathbb{R}_{max}^{m\times p}$ and $C \in \mathbb{R}_{max}^{p\times n}$ then

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij} \text{ and } [A \otimes C]_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes c_{kj}.$$

The relation " $\leq_{max}$ " defined in $\mathbb{R}_{max}^{m\times n}$ where $A \leq_{max} B \Leftrightarrow A \oplus B = B$ is a partial order on $\mathbb{R}_{max}^{m\times n}$.

Defined $\mathbb{R}_{\max}^{n} = \left\{\boldsymbol{x} = [x_1, x_2, \ldots, x_n]^T \mid x_j \in \mathbb{R}_{\max}, j = 1,2, \ldots, n\right\}$. The element of $\mathbb{R}_{\max}^{n}$ is called vector over $\mathbb{R}_{max}$.

Definition 1.1. *Given $A \in \mathbb{R}_{max}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{R}_{max}^{m}$. Subsolution of the system of max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is a vector $\boldsymbol{x}' \in \mathbb{R}_{max}^{n}$ that satisfies $A \otimes \boldsymbol{x}' \leq_{max} \boldsymbol{b}$.*

Definition 1.2. *A subsolution $\boldsymbol{x}^*$ of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is called the greatest subsolution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ if $\boldsymbol{x}' \leq_{max} \boldsymbol{x}^*$ for every subsolution $\boldsymbol{x}'$ of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$.*

Theorem 1.1. [4] *Given $A \in \mathbb{R}_{max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$, then $-x^*_j = \max_i(-b_i + a_{ij})$ for every $i \in \{1,2, \ldots, m\}$ and $j \in \{1,2, \ldots, n\}$.*

Theorem 1.2. [3] *Given $A \in \mathbb{R}_{max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$. $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has a solution if and only if $\boldsymbol{x}^*$ is a solution.*

2. Main Result

Base on Theorem 1.2, we can conclude that the existence of the solution of the system of max-plus linear equation $A \otimes \boldsymbol{x} = \boldsymbol{b}$ is determined by the greatest subsolution. Let $A \in \mathbb{R}_{\max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$. The case that we'll discuss is the solution of the system of max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ with more variables than more equations or $m < n$. The greatest subsolution is a candidate solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ that is vector $\boldsymbol{x}^*$ where

$$-\boldsymbol{x}^* = \begin{bmatrix} -x^*_1 \\ -x^*_2 \\ \vdots \\ -x^*_n \end{bmatrix} = \begin{bmatrix} \max_i(-b_i + a_{i1}) \\ \max_i(-b_i + a_{i2}) \\ \vdots \\ \max_i(-b_i + a_{in}) \end{bmatrix} = \begin{bmatrix} \max_i(a_{i1} - b_i) \\ \max_i(a_{i2} - b_i) \\ \vdots \\ \max_i(a_{in} - b_i) \end{bmatrix}$$

$$= \begin{bmatrix} \max\{a_{11} - b_1, a_{21} - b_2, \ldots, a_{m1} - b_m\} \\ \max\{a_{12} - b_1, a_{22} - b_2, \ldots, a_{m2} - b_m\} \\ \vdots \\ \max\{a_{1n} - b_1, a_{2n} - b_2, \ldots, a_{mn} - b_m\} \end{bmatrix}$$

Then, we define discrepancy matrix denoted by $D_{A,b}$ as folows

$$D_{A,b} = \begin{bmatrix} a_{11}-b_1 & a_{12}-b_1 & \dots & a_{1n}-b_1 \\ a_{21}-b_2 & a_{22}-b_2 & \dots & a_{2n}-b_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}-b_m & a_{m2}-b_m & \dots & a_{mn}-b_m \end{bmatrix}$$

Note that every $-x^*{}_j$ can be determined by taking the maximum of each column of $D_{A,b}$.

In order to predict the number of solutions of system $A \otimes \boldsymbol{x} = \boldsymbol{b}$, we define matrix $R_{A,b}$ that is reduced from $D_{A,b}$ as follows:

$$R_{A,b} = [r_{ij}] \text{ where } r_{ij} = \begin{cases} 1, & \text{if } d_{ij} = \text{maximum of column } j \\ 0, & \text{otherwise} \end{cases}$$

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Next, we will give the examples of the solution of the max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$ for $m < n$.

Example 2.1. Solve $A \otimes \boldsymbol{x} = \boldsymbol{b}$ if $A = \begin{bmatrix} 1 & 0 & 3 \\ \varepsilon & 4 & 2 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\boldsymbol{b} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

A quick calculation gives $D_{A,b} = \begin{bmatrix} 0 & -1 & 2 \\ \varepsilon & -2 & -4 \end{bmatrix}$ and $R_{A,b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Base on matrix $D_{A,b}$ we get $\boldsymbol{x}^* = [0,\ 1, -2]^T$. However, there is a row in $R_{A,b}$ that all entries are 0. It is the second row which means that there is no maximum in that row. It indicates that the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has no solution. We can verify that through the calculation as follows:

$$A \otimes \boldsymbol{x}^* = \begin{bmatrix} 1 & 0 & 3 \\ \varepsilon & 4 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \max\{1,1,1\} \\ \max\{\varepsilon, 5, 0\} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \leq \begin{bmatrix} 1 \\ 6 \end{bmatrix} = \boldsymbol{b}$$

So, $\boldsymbol{x}^*$ is just the greates subsolution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ but not the solution.

Example 2.2. Solve $A \otimes \boldsymbol{x} = \boldsymbol{b}$ if $A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 7 & 8 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, dan $\boldsymbol{b} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

A quick calculation gives $D_{A,b} = \begin{bmatrix} -2 & -2 & -3 \\ -2 & 1 & 2 \end{bmatrix}$ and $R_{A,b} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Base on matrix $D_{A,b}$ we get $\boldsymbol{x}^* = [2, -1, -2]^T$. Next, we will verify whether $\boldsymbol{x}^*$ is a solution or not.

$$A \otimes \boldsymbol{x}^* = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 7 & 8 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} \max\{4, 1, -1\} \\ \max\{6,6,6\} \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \boldsymbol{b}$$

We can see that $\boldsymbol{x}^*$ is indeed the solution of $A \otimes \boldsymbol{x} = \boldsymbol{b}$**.** But, there is more than one 1 in the second row of $R_{A,b}$. In other words, there is more than one maximum in that row. It indicates that the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has an infinite number of solutions. Base on Definition 1.2, we know that the elements of $\boldsymbol{x}^*$ are the upper bounds. So, the elements of vector $\boldsymbol{x}$ in this example must satisfy $x_1 \leq 2$, $x_2 \leq -1$ and $x_3 \leq -2$. On the first row of $R_{A,b}$, the maximum is in the first column then $x_1 = 2$. On the second row, the maximum is in the first, second and third column then there are three possible ways with either $x_1 = 2$, $x_2 = -1$ or $x_3 = -2$. If we change the value of x_1 then it will change the equation in the first row. So now as long as $x_2 \leq -1$ and $x_3 \leq -2$, the first and second equation will always be true. Therefore, every vector $\boldsymbol{x} = [2,\ a,\ b]^T$ where $a \le -1$ and $b \le -2$ is also a solution. So, the system of $A \otimes \boldsymbol{x} = \boldsymbol{b}$ in this example has an infinite number of solutions.

Matrix $D_{A,b}$ and $R_{A,b}$ play role in determining the characteristics of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$. Now, we will give the theorem about the exixtence of the solution of the system of max-plus linear equations $A \otimes \boldsymbol{x} = \boldsymbol{b}$.

Theorem 2.1. [1] *Given the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ where $A \in \mathbb{R}_{max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$.*
1. If there is a zero-row in matrix $R_{A,b}$ then the system has no solution.
2. If there is at least one 1 in each row of $R_{A,b}$, then $\boldsymbol{x}^*$ is the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$.

Proof.
1. Without lost of generality, suppose the zero-row of $R_{A,b}$ is the k^{th} and let $\boldsymbol{x}^*$ is the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$, then $-x^*_j \ge \max_i(-b_i + a_{ij}) > -b_k + a_{kj}$. Thus, $-x^*_j > -b_k + a_{kj} \Leftrightarrow a_{kj} + x^*_j < b_k, \forall j$. Hence, $\boldsymbol{x}^*$ does not satisfy the k^{th} equation. It contradicts with $\boldsymbol{x}^*$ is the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$. So, the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ has no solution.
2. We will proof the contrapositive. Suppose $\boldsymbol{x}^*$ is not the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$. By Teorema 1.1, $-x^*_j \ge -b_k + a_{kj}, \forall k,j$. Thus, $\max_j(a_{kj} + x^*_j) \le b_k$. If $\boldsymbol{x}^*$ is not the solution of the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ then there is k such that $\max_j(a_{kj} + x^*_j) < b_k$. This is equivalent to $-x^*_j > -b_k + a_{kj}, \forall j$. Since $-x^*_j = \max(-b_l + a_{lj})$ for some l, then there is no element in the k^{th} row of $R_{A,b}$ thas is 1. ■

In order to determine the uniqueness of the solution of system of max-plus linear equations, we give the definition as follows

Definition 2.1. *The 1 in a row of $R_{A,b}$ is a **variable-fixing entry** if either*
*1. It is the only **1** in that row (a **lone-one)**, or*
*2. It is in the same column as a **lone-one.***
*The remaining **1s** are called **slack entries.***

The 1s that are circled in the above examples are the variable-fixing entries.

Theorem 2.2. [4] *Given the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ where $A \in \mathbb{R}_{max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$ and the solution to the system exist.*
1. *If each row of $R_{A,b}$ has a **lone one**, then the solution of the systemis unique*
2. *If there are **slack entries** in $R_{A,b}$, then the system has infinite solutions.*

Corollary 2.1. *Given the system $A \otimes \boldsymbol{x} = \boldsymbol{b}$ where $A \in \mathbb{R}_{max}^{m\times n}$ with the entries of each column are not all equal ε and $\boldsymbol{b} \in \mathbb{R}^m$ and $m < n$. If there is no zero-row in $R_{A,b}$ then there are infinite solutions of the system.*

Proof. Matrix $R_{A,b}$ has no zero-row so there is at least one **1** in each row of $R_{A,b}$. Suppose that the solution of the system is unique then there is a ***lone one*** in each row of $R_{A,b}$. Meanwhile, $m < n$ which means that there are more variables than more equations in that system. Hence, there must be *slack entries* in $R_{A,b}$. This contradicts with the solution of the system is unique. So, there are infinite solutions of the system. ■

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