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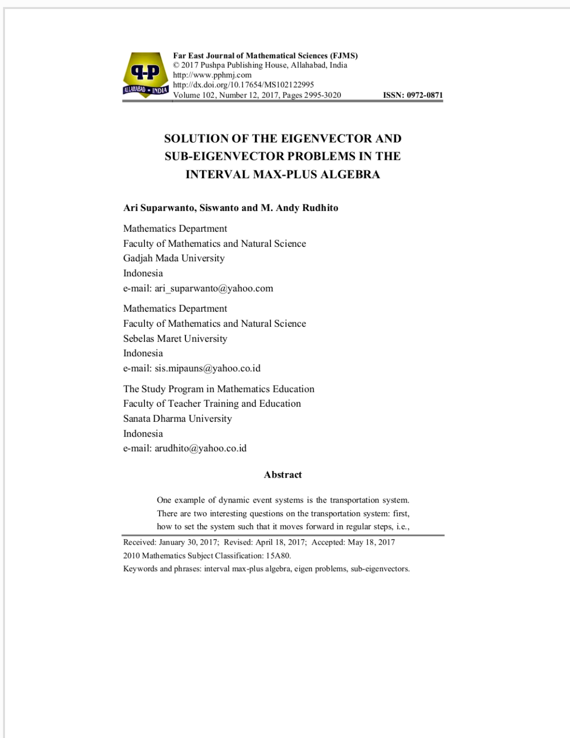

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SOLUTION OF THE EIGENVECTOR AND SUB-EIGENVECTOR PROBLEMS IN THE INTERVAL MAX-PLUS ALGEBRA

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Abstract

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Abstract

One example of dynamic event systems is the transportation system. There are two interesting questions on the transportation system: first, how to set the system such that it moves forward in regular steps, i.e.,

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for a given constant λ, the time interval between any initial points of consecutive cycles on every crossroad is λ? Second, suppose that there is a given schedule for the system and the time interval between two consecutive starting points of any two consecutive tasks does not exceed a certain value μ, is it possible to start the system in such a way that the schedule is kept? The first problem is closely related to eigenvector problems, and the second problem is related to the sub-eigenvector problems.

This paper gives an overview of eigen problems (eigenvalues and eigenvectors) in interval max-plus algebra related to eigen problems with finite solution. We discuss about the criteria of finite eigenvectors existence and the description of the space of all finite eigenvectors of any square matrices. Furthermore, this paper also discusses about sub-eigenvector problems.

1. Introduction

This study is started from the problems of road transportation. One of which is the coordination of traffic lights at the crossroads (Pesko et al. [8]). Problems and mathematical models of traffic light coordination are associated with eigenvalue, eigenvector and sub-eigenvector problems (Turek and Turek [12] and Cuninghame-Green [5]). The problems of eigenvalues, eigenvectors and sub-eigenvectors can be explained in the case of max-plus algebra (Butkovic [2]). To determine the eigenvalues and eigenvectors over max-plus algebra, one can determine with the maximum average method, the method of power or the application of linear programming (Chung [4]).

Max-plus algebra is the set $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$, where $\mathbb{R}$ is the set of all real numbers, $\varepsilon = -\infty$ equipped with operations $\oplus$ (maximum) and $\otimes$ (sum) (Bacceli et al. [1]). Max-plus algebra is an idempotent semi field. From the set $\mathbb{R}_\varepsilon$, one can form the set of $m \times n$ matrices whose elements are in $\mathbb{R}_\varepsilon$, which later is called the *set of matrices on max-plus algebra* and is denoted by $\mathbb{R}_\varepsilon^{m \times n}$ (Butkovic [2]). For $n = 1$, we obtain the set of vectors on max-plus algebra and written by $\mathbb{R}_{\varepsilon}^{m}$. Furthermore, if $m = n$, then the set $\mathbb{R}_{\varepsilon}^{n\times n}$ together with two binary operations $\oplus$ (maximum) and $\otimes$ (sum) is an idempotent semiring. The definition and some properties of eigenvalues and eigenvectors can be explained on the algebraic structure $\mathbb{R}_{\varepsilon}^{n\times n}$. Graph theory which discusses the definition and properties of digraphs, communication graphs, paths, cycles, elementary cycles and weight of paths, loop, strongly connected digraph is used to define irreducible and reducible matrices, introduce the theory of maximum average cycle, the matrix R-astic, transitive closure and definite matrix in max-plus algebra (Butkovic [2], Carre [3] and Tam [11]). Subspace theory over max-plus algebra is based on the theory of semimodule and sub-semimodule (Golan [7]).

It is known that a measurement is never 100% accurate. As a result, the values of measurement $\tilde{x}_i$ are generally different from the actual value x_i. In particular, in the case of uncertainty interval, after the measurement resulting $\tilde{x}_i$, the information obtained is that the real value x_i of the measurement process is contained in the interval $\mathbb{X}_i = [\tilde{x}_i - \Delta, \tilde{x}_i + \Delta]$. Therefore, it is possible to provide the time course of a particular process in a given interval.

Max-plus algebra can be generalized into interval max-plus algebra, where we can also discuss matrix, graph, eigenvalues and eigenvectors of a matrix in interval max-plus algebra (Rudhito [9]). However, eigenvalues and eigenvectors of matrix in interval max-plus algebra have been discussed only for irreducible matrices. Therefore, it is still possible to discuss, in general, about the problem of eigenvalues and eigenvectors as well as the bases for the eigen space of the matrix in interval max-plus algebra, through a different approach done before (Siswanto et al. [10]).

To resolve network problems with activity time interval, max-plus algebra has been generalized into an interval max-plus algebra. Interval max-plus algebra is the set $I(\mathbb{R})_{\varepsilon} = \{x = [\underline{x}, \overline{x}] \mid \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \le \overline{x}\} \cup \{\varepsilon\}$, with $\varepsilon = [\varepsilon, \varepsilon]$, equipped with two binary operations $\overline{\oplus}$ and $\overline{\otimes}$ with $x \,\overline{\oplus}\, y =$

$[\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $x \overline{\otimes} y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ for every $x, y \in I(\mathbb{R})_{\varepsilon}$. It has also been discussed about the matrix over interval max-plus algebra, that is $I(\mathbb{R})_{\varepsilon}^{n \times n}$.

Based on the idea above, we will try to extend the concepts in algebra max-plus into the interval max-plus algebra. These concepts include criteria of existence of finite eigenvectors and the description of all finite eigenvectors of square matrices over interval max-plus algebra. Furthermore, sub-eigenvector problems in the interval max-plus algebra will also be investigated.

2. Main Result

Max-plus algebra is the set $\mathbb{R}_{\varepsilon} = \mathbb{R} \cup \{\varepsilon\}$, of which $\mathbb{R}$ is the set of all real numbers and $\varepsilon = -\infty$ that is equipped with two operations $\oplus$ (maximum) and $\otimes$ (sum). From this, one can form a set of matrices of size $m \times n$ over $\mathbb{R}_{\varepsilon}$, denoted by $\mathbb{R}_{\varepsilon}^{m \times n}$. Moreover, whenever $m = n$, the set $\mathbb{R}_{\varepsilon}^{n \times n}$ equipped with operations $\oplus$ and $\otimes$ forms an idempotent semiring.

By a closed interval x in $\mathbb{R}_{\varepsilon}$, we mean a subset of $\mathbb{R}_{\varepsilon}$ in the form $x = [\underline{x}, \overline{x}] = \{x \in \mathbb{R}_{\varepsilon} \mid \underline{x} \le x \le \overline{x}\}$. The interval x in $\mathbb{R}_{\varepsilon}$ is known as *interval max-plus*. A number $x \in \mathbb{R}_{\varepsilon}$ can be expressed as interval $[x, x]$.

By $I(\mathbb{R})_{\varepsilon}$, we mean the set $\{x = [\underline{x}, \overline{x}] \underline{x}, \overline{x} \in \mathbb{R}, \varepsilon < \underline{x} \le \overline{x}\} \cup \{\varepsilon\}$, with $\varepsilon = [\varepsilon, \varepsilon]$. On the set $I(\mathbb{R})_{\varepsilon}$, we define two operations $\overline{\oplus}$ and $\overline{\otimes}$ with $x \overline{\oplus} y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$ and $x \overline{\otimes} y = [\underline{x} \otimes \underline{y}, \overline{x} \otimes \overline{y}]$ for every $x, y \in I(\mathbb{R})_{\varepsilon}$. The set $I(\mathbb{R})_{\varepsilon}$ equipped with the operations $\overline{\oplus}$ and $\overline{\otimes}$ is a commutative idempotent semiring with neutral element $\varepsilon = [\varepsilon, \varepsilon]$ and the unit element $\overline{0} = [0, 0]$. The semiring $(I(\mathbb{R})_{\varepsilon}; \overline{\oplus}, \overline{\otimes})$ then is called *interval max-plus algebra* and denoted by $I(\mathbb{R})_{\max} = (I(\mathbb{R})_{\varepsilon}; \overline{\oplus}, \overline{\otimes})$.

The set of all matrices of size $m \times n$ over $I(\mathbb{R})_\varepsilon$ is denoted by $I(\mathbb{R})_\varepsilon^{m\times n}$, i.e.,

$$I(\mathbb{R})_\varepsilon^{m\times n} = \{A = [A_{ij}] \mid A_{ij} \in I(\mathbb{R})_\varepsilon;\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n\}.$$

All the matrices which belong to $I(\mathbb{R})_\varepsilon^{m\times n}$ are called *matrices over interval max-plus algebra*. The set $I(\mathbb{R})_\varepsilon^{n\times n}$ equipped with operations $\overline{\oplus}$ and $\overline{\otimes}$ forms an idempotent semiring and denoted by $I(\mathbb{R})_{\max}^{n\times n} = (I(\mathbb{R})_\varepsilon^{n\times n}; \overline{\oplus}, \overline{\otimes})$, whereas $I(\mathbb{R})_\varepsilon^{n\times n}$ is a semimodule over $I(\mathbb{R})_\varepsilon$.

For a matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$, $\underline{A} = [\underline{A}_{ij}] \in \mathbb{R}_\varepsilon^{m\times n}$ and $\overline{A} = [\overline{A}_{ij}] \in \mathbb{R}_\varepsilon^{m\times n}$ denote lower and upper bound matrices of the interval matrix A, respectively.

Given a matrix $A \in I(\mathbb{R})_\varepsilon^{m\times n}$. Let $\underline{A}$ and $\overline{A}$ be lower and upper bound matrices of the interval matrix A, respectively. We define the interval matrix of A, namely $[\underline{A}, \overline{A}] = \{A \in \mathbb{R}_{\max}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\}$ and $I(\mathbb{R}_\varepsilon^{m\times n})_b = \{[\underline{A}, \overline{A}] \mid A \in I(\mathbb{R})_\varepsilon^{m\times n}\}$.

For $\alpha \in I(\mathbb{R})_\varepsilon$, $[\underline{A}, \overline{A}], [\underline{B}, \overline{B}] \in I(\mathbb{R}_\varepsilon^{m\times n})_b$, we define two operations $\overline{\otimes}$ and $\overline{\oplus}$ by:

(1) $\alpha \,\overline{\otimes}\, [\underline{A}, \overline{A}] = [\underline{\alpha} \otimes \underline{A}, \overline{\alpha} \otimes \overline{A}]$,

(2) $[\underline{A}, \overline{A}] \,\overline{\oplus}\, [\underline{B}, \overline{B}] = [\underline{A} \oplus \overline{A}, \underline{B} \oplus \overline{B}]$.

For $[\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{m\times n})_b$, $[\underline{B}, \overline{B}] \in I(\mathbb{R}_\varepsilon^{k\times n})_b$, we define $[\underline{A}, \overline{A}] \,\overline{\otimes}\, [\underline{B}, \overline{B}] := [\underline{A} \otimes \overline{A}, \underline{B} \otimes \overline{B}]$.

The set $I(\mathbb{R}_\varepsilon^{n\times n})_b$ equipped with operations $\overline{\oplus}$ and $\overline{\otimes}$ forms an idempotent semiring and is denoted by $I(\mathbb{R}_{\max}^{n\times n})_b = (I(\mathbb{R}_\varepsilon^{n\times n})_b; \overline{\oplus}, \overline{\otimes})$, whereas $I(\mathbb{R}_\varepsilon^{n\times n})_b$ is semimodule over $I(\mathbb{R})_\varepsilon$.

The semiring $I(\mathbb{R})_{\max}^{n\times n} = (I(\mathbb{R})_{\varepsilon}^{n\times n}; \overline{\oplus}, \overline{\otimes})$ is isomorphic with the semiring $I(\mathbb{R}_{\max}^{n\times n})_b = (I(\mathbb{R}_{\varepsilon}^{n\times n})_b; \overline{\oplus}, \overline{\otimes})$ by the isomorphism $f : I(\mathbb{R})_{\varepsilon}^{n\times n} \to I(\mathbb{R}_{\varepsilon}^{n\times n})_b$, $f(A) = [\underline{A}, \overline{A}]$, $\forall A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$. The semimodule $I(\mathbb{R})_{\varepsilon}^{n\times n}$ over $I(\mathbb{R})_{\varepsilon}$ is isomorphic with semimodule $I(\mathbb{R}_{\varepsilon}^{n\times n})_b$ over $I(\mathbb{R})_{\varepsilon}$. Thus, for each interval matrix A, one can always determine the matrix interval $[\underline{A}, \overline{A}]$ and vice versa, and for each interval matrix $[\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon}^{n\times n})_b$ with $\underline{A}, \overline{A} \in \mathbb{R}_{\varepsilon}^{n\times n}$, one can also determine the interval matrix $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$, $[\underline{A}_{ij}, \overline{A}_{ij}] \in I(\mathbb{R})_{\varepsilon}$ for every i and j. Thus, the matrix $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ can be viewed as an interval matrix $[\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon}^{n\times n})_b$. An interval matrix $[\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon}^{n\times n})_b$ is called an *interval matrix* corresponding to the matrix $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ and is denoted by $A \approx [\underline{A}, \overline{A}]$. As a result of the above isomorphism, we have: $\alpha \overline{\otimes} A \approx [\underline{\alpha} \otimes \underline{A}, \overline{\alpha} \otimes \overline{A}]$, $A \overline{\oplus} B \approx [\underline{A} \oplus \underline{B}, \overline{A} \oplus \overline{B}]$ and $A \overline{\otimes} B \approx [\underline{A} \otimes \underline{B}, \overline{A} \otimes \overline{B}]$.

Let $I(\mathbb{R})_{\varepsilon}^{n}$ be the set $\{x = [x_1, x_2, ..., x_n]^T \mid x_i \in I(\mathbb{R})_{\varepsilon};\ i = 1, 2, ..., n\}$. The set $I(\mathbb{R})_{\varepsilon}^{n}$ can be viewed as a set $I(\mathbb{R})_{\varepsilon}^{n\times 1}$. The elements of $I(\mathbb{R})_{\varepsilon}^{n}$ are called *interval vectors* in $I(\mathbb{R})_{\varepsilon}$. An interval vector that is corresponding to the interval vector x is $x \approx [\underline{x}, \overline{x}]$.

Furthermore, we present the concept of interval weighted directed graph. Given a directed graph $D = (N, E)$ with $N = \{1, 2, ..., n\}$. A directed graph is said to be *weighted interval* if each arc $(j, i) \in E$ can be associated with a closed interval of real numbers $A_{ij} \in (I(\mathbb{R})_{\varepsilon} - \{[\varepsilon, \varepsilon]\})$. Interval of real number A_{ij} is called *interval weight* of arc (j, i) and is denoted by $w(i, j)$.

We define the graph precedent interval (graph communication interval) of the matrix $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ as an interval weighted directed graph $D_A =$

(N, E) with $N = \{1, 2, ..., n\}$ and $E = \{(j, i) | w(i, j) = A_{ij} \neq [\varepsilon, \varepsilon]\}$. For every interval weighted directed graph $D_A = (N, E)$, it can always be defined as a matrix $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ called *matrix of interval weighted graph* with $A_{ij} = \begin{cases} w(j, i), & (i, j) \in E \\ [\varepsilon, \varepsilon], & (i, j) \notin E. \end{cases}$

A directed graph $D = (N, E)$ is said to be *strongly connected* if u can be reached from v for all $u, v \in N$.

Suppose that $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ and D_A is a weighted directed graph corresponding to A. The matrix A is irreducible if D_A is strongly connected. Otherwise, A is said to be *reducible*.

We discuss the eigenvalues and eigenvectors of matrices over interval max-plus algebra.

Given a matrix $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$. The interval scalar $\lambda \in I(\mathbb{R})_{\varepsilon}$ is called *eigenvalue* of interval matrix A if there exists an interval vector $v \in I(\mathbb{R})_{\varepsilon}^{n}$ with $v \neq \varepsilon_{n+1}$ satisfying $A \overline{\otimes} v = \lambda \overline{\otimes} v$. Vector v is called *eigenvector* of interval matrix A corresponding to the eigenvalue λ.

For a given $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$, the problem to determine $x \in I(\mathbb{R})_{\varepsilon}^{n}$, $x \neq \varepsilon_{n+1}$ and $\lambda \in I(\mathbb{R})_{\varepsilon}$ such that $A \overline{\otimes} x = \lambda \overline{\otimes} x$ is called the *problem* of eigenvalues and eigenvectors in interval max-plus algebra.

Eigenvalue and eigenvectors of an irreducible matrix

Next, we will discuss eigenvalues and eigenvectors of an irreducible matrix.

Definition 1. Given a matrix $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ and $\lambda = [\underline{\lambda}, \overline{\lambda}] \in I(\mathbb{R})_{\varepsilon}$, we define:

(a) $V(A, \lambda) = \{x \in I(\mathbb{R})_\varepsilon^n \mid x \approx [\underline{x}, \overline{x}] \ni \underline{x} \in V(\underline{A}, \underline{\lambda}); \overline{x} \in V(\overline{A}, \overline{\lambda})\}$, where for $A \in \mathbb{R}_\varepsilon^{n \times n}$, $V(A, \lambda) = \{x \in \mathbb{R}_\varepsilon^n \mid A \otimes x = \lambda \otimes x\}$,

(b) $\Lambda(A) = \{\lambda = [\underline{\lambda}, \overline{\lambda}] \in I(\mathbb{R})_\varepsilon \mid V(\underline{A}, \underline{\lambda}) \neq \{\varepsilon\}; V(\overline{A}, \overline{\lambda}) \neq \overline{\varepsilon}\}$,

(c) $V(A) = \bigcup_{\lambda \in \Lambda(A)} V(A, \lambda)$,

(d) $V^+(A, \lambda) = V(A, \lambda) \bigcap I(\mathbb{R})^n$,

(e) $V^+(A) = V(A) \bigcap I(\mathbb{R})^n$.

Theorem 1. *If* $A, B \in I(\mathbb{R})_\varepsilon^{n \times n}$, $\alpha = [\underline{\alpha}, \overline{\alpha}] \in I(\mathbb{R})$, $[\underline{\lambda}, \overline{\lambda}], [\underline{\mu}, \overline{\mu}] \in I(\mathbb{R})_\varepsilon$, *then*

(a) $V(\alpha \overline{\otimes} A) = V(A)$,

(b) $\Lambda(\alpha \overline{\otimes} A) = \alpha \overline{\otimes} \Lambda(A)$,

(c) $V(A, \lambda) \bigcap V(B, \mu) \subseteq V(A \overline{\otimes} B, \lambda \overline{\otimes} \mu)$,

(d) $V(A, \lambda) \bigcap V(B, \mu) \subseteq V(A \overline{\oplus} B, \lambda \overline{\oplus} \mu)$.

Proof. (a) By Definition 1, to prove that $V(\alpha \overline{\otimes} A) = V(A)$, we sufficiently prove that $V(A, \lambda) = V(\alpha \overline{\otimes} A, \lambda)$, for each $\lambda \in \Lambda(A)$. We know that

(i) $\bigcup_{\underline{\lambda} \in \Lambda(\underline{A})} V(\underline{\alpha} \otimes \underline{A}, \underline{\lambda}) = V(\underline{\alpha} \otimes \underline{A}) = V(\underline{A}) = \bigcup_{\underline{\lambda} \in \Lambda(\underline{A})} V(\underline{A}, \underline{\lambda})$,

(ii) $\bigcup_{\overline{\lambda} \in \Lambda(\overline{A})} V(\overline{\alpha} \otimes \overline{A}, \overline{\lambda}) = V(\overline{\alpha} \otimes \overline{A}) = V(\overline{A}) = \bigcup_{\overline{\lambda} \in \Lambda(\overline{A})} V(\overline{A}, \overline{\lambda})$.

Therefore, $V(\underline{\alpha} \otimes \underline{A}, \underline{\lambda}) = V(\underline{A}, \underline{\lambda})$ and $V(\overline{\alpha} \otimes \overline{A}) = V(\overline{A}, \overline{\lambda})$. Take $x \in V(A, \lambda)$, then $x \in I(\mathbb{R})_\varepsilon^n$, $x \approx [\underline{x}, \overline{x}]$, where $\underline{x} \in V(\underline{A}, \underline{\lambda})$ and $\overline{x} \in V(\overline{A}, \overline{\lambda})$. Consequently, $\underline{x} \in V(\underline{\alpha} \otimes \underline{A}, \underline{\lambda})$ and $\overline{x} \in V(\overline{\alpha} \otimes \overline{A})$, so $x \in V(\alpha \overline{\otimes} A, \lambda)$. Conversely, take $x \in V(\alpha \overline{\otimes} A, \lambda)$, then $\underline{x} \in V(\underline{\alpha} \otimes \underline{A}, \underline{\lambda})$

and $\overline{x} \in V(\overline{\alpha} \otimes \overline{A}, \overline{\lambda})$. Consequently, $\underline{x} \in V(\underline{A}, \underline{\lambda})$ and $\overline{x} \in V(\overline{A}, \overline{\lambda})$, so $x \in V(A, \lambda)$. Therefore, $V(A, \lambda) = V(\alpha \,\overline{\otimes}\, A, \lambda)$. Hence, $V(\alpha \,\overline{\otimes}\, A) = V(A)$.

(b) Suppose $x \approx [\underline{x}, \overline{x}] \in V(A)$. Then $\underline{x}$ and $\overline{x}$ are the eigenvectors that correspond to the eigenvalues $\lambda = [\underline{\lambda}, \overline{\lambda}]$, so we have $\underline{x} \in V(\underline{A}, \underline{\lambda})$ and $\overline{x} \in V(\overline{A}, \overline{\lambda})$. Therefore,

(i)

$$\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{x} \Leftrightarrow \underline{\alpha} \otimes (\underline{A} \otimes \underline{x}) = \underline{\alpha} \otimes (\underline{\lambda} \otimes \underline{x})$$

$$\Leftrightarrow (\underline{\alpha} \otimes \underline{A}) \otimes \underline{x} = (\underline{\alpha} \otimes \underline{A}) \otimes \underline{x},$$

(ii)

$$\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{x} \Leftrightarrow \overline{\alpha} \otimes (\overline{A} \otimes \overline{x}) = \overline{\alpha} \otimes (\overline{\lambda} \otimes \overline{x})$$

$$\Leftrightarrow (\overline{\alpha} \otimes \overline{A}) \otimes \overline{x} = (\overline{\alpha} \otimes \overline{\lambda}) \otimes \overline{x}.$$

Thus, the eigenvalues of matrices $\underline{\alpha} \otimes \underline{A}$ and $\overline{\alpha} \otimes \overline{A}$, respectively, are $\underline{\alpha} \otimes \underline{\lambda}$ and $\overline{\alpha} \otimes \overline{\lambda}$. By Definition 1, $\Lambda(\alpha \,\overline{\otimes}\, A) = \alpha \,\overline{\otimes}\, \Lambda(A)$.

(c) Let $x \in V(A, \lambda) \cap V(B, \mu)$, i.e., $x \in V(A, \lambda)$ and $x \in V(B, \mu)$. Therefore, $A \,\overline{\otimes}\, x = \lambda \,\overline{\otimes}\, x$ and $B \,\overline{\otimes}\, x = \mu \,\overline{\otimes}\, x$ or $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{x}$; $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{x}$ and $\underline{B} \otimes \underline{x} = \underline{\mu} \otimes \underline{x}$; $\overline{B} \otimes \overline{x} = \overline{\mu} \otimes \overline{x}$. Furthermore, $(\underline{A} \otimes \underline{B}) \otimes \underline{x} = \underline{A} \otimes (\underline{B} \otimes \underline{x}) = \underline{A} \otimes (\underline{\mu} \otimes \underline{x}) = (\underline{A} \otimes \underline{\mu}) \otimes \underline{x} = (\underline{\mu} \otimes \underline{A}) \otimes \underline{x} = \underline{\mu} \otimes (\underline{A} \otimes \underline{x}) = \underline{\mu} \otimes (\underline{\lambda} \otimes \underline{x}) = (\underline{\mu} \otimes \underline{\lambda}) \otimes \underline{x}$. Likewise, we also have $(\overline{A} \otimes \overline{B}) \otimes \overline{x} = (\overline{\mu} \otimes \overline{\lambda}) \otimes \overline{x}$. Therefore, $x \in V(A \,\overline{\otimes}\, B, \lambda \,\overline{\otimes}\, \mu)$, so we have $V(A, \lambda) \cap V(B, \mu) \subseteq V(A \,\overline{\otimes}\, B, \lambda \,\overline{\otimes}\, \mu)$.

(d) Let $x \in V(A, \lambda) \cap V(B, \mu)$, i.e., $x \in V(A, \lambda)$ and $x \in V(B, \mu)$. Therefore, $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{x}$; $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{x}$ and $\underline{B} \otimes \underline{x} = \underline{\mu} \otimes \underline{x}$; $\overline{B} \otimes \overline{x} = \overline{\mu} \otimes \overline{x}$. Furthermore, $(\underline{A} \oplus \underline{B}) \otimes \underline{x} = (\underline{A} \otimes \underline{x}) \oplus (\underline{B} \otimes \underline{x}) = (\underline{\lambda} \otimes \underline{x}) \oplus (\underline{\mu} \otimes \underline{x})$

$= (\underline{\lambda} \oplus \underline{\mu}) \otimes \underline{x}$. Likewise, we also have that $(\overline{A} \oplus \overline{B}) \otimes \overline{x} = (\overline{\lambda} \oplus \overline{\mu}) \otimes \overline{x}$. Therefore, $x \in V(A \overline{\otimes} B, \lambda \overline{\otimes} \mu)$, so we have that $V(A, \lambda) \cap V(B, \mu) \subseteq V(A \overline{\otimes} B, \lambda \overline{\otimes} \mu)$. □

We define the critical points of the matrix $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ as follows.

Definition 2. Suppose $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon}^{n \times n})_b$ and $N = \{1, 2, ..., n\}$. By $E(A)$, we mean $E(A) = E(\underline{A}) \cap E(\overline{A})$, where for $A \in \mathbb{R}_{\varepsilon}^{n \times n}$, $E(A) = \{i \in N \mid \exists \sigma = (i = i_1, i_2, ..., i_k, i_1) \text{ in } D_A \ni \mu(\sigma, A) = \lambda(A)\}$. The elements of $E(A)$ are called the *critical points* of A.

The cycle σ is called *critical cycle* if $\mu(\sigma, \underline{A}) = \underline{\lambda}(\underline{A})$ and $\mu(\sigma, \overline{A}) = \overline{\lambda}(\overline{A})$. The points in N and union of the set of arcs of all critical cycles form a directed graph $C(A)$ and it is said to be *critical directed graph* of A. The following lemma talks about the critical directed graph.

Lemma 1. *Suppose* $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ *and* $C(A)$ *is the critical directed graph of A. Then all the cycles in* $C(A)$ *are critical.*

Proof. Suppose $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ and $A \approx [\underline{A}, \overline{A}]$. It is already known that $C(A)$ is a critical directed graph of A. Let $C(\underline{A})$ and $C(\overline{A})$ be the critical directed graphs of $\underline{A}$ and $\overline{A}$, respectively. Then all the cycles in $C(\underline{A})$ and $C(\overline{A})$ are critical. Therefore, all the cycles in $C(A)$ are critical. □

Two points i and j in $C(A)$ are said to be *equivalent* if two points are in the same cycle and denoted by $i \sim j$. The relation is an equivalence relation in $E(A)$.

Lemma 2. *Let* $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$, *and* $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon}^{n \times n})_b$. *If* $\lambda(A) = [\varepsilon, \varepsilon]$, *then* $\Lambda(A) = \{[\varepsilon, \varepsilon]\}$ *and the eigenvectors of A are vectors*

$[x_1, x_2, ..., x_n]^T \in (\mathbb{R})_\varepsilon^n$ *such that* $x_j = [\varepsilon, \varepsilon]$ *if the jth column of A is not equal to the vector* $[[\varepsilon, \varepsilon], [\varepsilon, \varepsilon], ..., [\varepsilon, \varepsilon]]^T$, $j \in N$.

Proof. It is known that $\lambda(A) = [\varepsilon, \varepsilon]$. For the lower bound matrix $\underline{A}$ of A, $\underline{\lambda}(\underline{A}) = \varepsilon$. Thus, we have $\Lambda(\underline{A}) = \{\varepsilon\}$ and eigenvectors of the matrix $\underline{A}$ as $[\underline{x}_1, \underline{x}_2, ..., \underline{x}_n]^T \in (\mathbb{R})_\varepsilon^n$, so that $\underline{x}_j = \varepsilon$, when the jth column of the matrix $\underline{A}$ is not the same as $[\varepsilon, \varepsilon, ..., \varepsilon]^T$, $j \in N$. Likewise, for the upper bound matrix $\overline{A}$, we have the same. Therefore, we obtain $\Lambda(A) = \{[\varepsilon, \varepsilon]\}$ and eigenvectors of the matrix A as $[x_1, x_2, ..., x_n]^T \approx [(\underline{x}_1, \underline{x}_2, ..., \underline{x}_n)^T, (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)^T]$, so that $x_j = [\varepsilon, \varepsilon]$, when the jth column of the matrix A is not equal to the vector $[[\varepsilon, \varepsilon], [\varepsilon, \varepsilon], ..., [\varepsilon, \varepsilon]]^T$, $j \in N$. □

The above lemma describes the eigenvector matrix A, if $\lambda(A) = [\varepsilon, \varepsilon]$. Furthermore, we will discuss for $[\varepsilon, \varepsilon] < [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})] = \lambda(A)$. Suppose $A \in I(\mathbb{R})_\varepsilon^n$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$ and $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})] > [\varepsilon, \varepsilon]$ is an eigenvalue of matrix A. Because $\underline{\lambda}(\underline{A})$ and $\overline{\lambda}(\overline{A})$ are eigenvalues of matrices $\underline{A}$ and $\overline{A}$ with $\underline{\lambda}(\underline{A}) > \varepsilon$ and $\overline{\lambda}(\overline{A}) > \varepsilon$, respectively, we can determine matrices $\Gamma(\underline{A}_\lambda) = (\underline{g}_{ij})$ and $\Gamma(\overline{A}_\lambda) = (\overline{g}_{ij})$, respectively.

Theorem 2. *Suppose* $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$ *and* $\lambda(A)$ *is an eigenvalue of matrix A. Let* $[\varepsilon, \varepsilon] < \lambda(A)$. *If* Γ_i *is an ith column of matrix* $\Gamma(A, \lambda)$ *such that the lower bound of the ith entry of* Γ_i *is equal to* 0 *and the upper bound vector of* Γ_i *is an eigenvector of* $\overline{A}$, *then* Γ_i *is an eigenvector of A corresponding to the eigenvalue* $\lambda(A)$.

Proof. It is known that $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$ and $\lambda(A) = [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})]$ is an eigenvalue of matrix A, where $\underline{\lambda}(\underline{A})$ and $\overline{\lambda}(\overline{A})$

are eigenvalues of matrices $\underline{A}$ and $\overline{A}$, respectively. Because $[\varepsilon, \varepsilon] < [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})] = \lambda(A)$, then $\underline{\lambda}(\underline{A}) > \varepsilon$ and $\overline{\lambda}(\overline{A}) > \varepsilon$, so we can determine the matrices $\Gamma(\underline{A}_\lambda) = (\underline{g}_{ij})$ and $\Gamma(\overline{A}_\lambda) = (\overline{g}_{ij})$. Suppose $\underline{g}_k$ and $\overline{g}_k$, $k = 1, 2, ..., n$, respectively, the columns of the matrices $\Gamma(\underline{A}_\lambda)$ and $\Gamma(\overline{A}_\lambda)$. Based on the case in max-plus algebra, from each of $\Gamma(\underline{A}_\lambda)$ and $\Gamma(\overline{A}_\lambda)$, we obtain at most n eigenvectors corresponding to $\underline{A}_\lambda$ and $\overline{A}_\lambda$. The eigenvectors are the columns of $\Gamma(\underline{A}_\lambda)$ and $\Gamma(\overline{A}_\lambda)$ where the main diagonal is equal to 0. We form matrix $\Gamma(A_\lambda)$ with columns defined as follows:

(a) If for some k, a pair of $\underline{g}_k$ and $\overline{g}_k$ satisfies $\underline{g}_k \leq \overline{g}_k$, then we determine the kth column as an interval vector $g_k \approx [\underline{g}_k, \overline{g}_k]$.

(b) If for some k, a pair of $\underline{g}_k$ and $\overline{g}_k$ does not satisfy $\underline{g}_k \leq \overline{g}_k$, then we determine $\overline{g}_k^* = \delta \otimes \overline{g}_k$ with $\delta = \max_i((\underline{g}_k)_i - (\overline{g}_k)_i)$, $i = 1, 2, ..., n$ and the kth column as an interval vector $g_k \approx [\underline{g}_k, \overline{g}_k^*]$. □

By Theorem 2, we have a method to determine the eigenvectors of an irreducible matrix. In the subsequent discussion, it will be shown that $\lambda(A)$ is the largest eigenvalue and called the *main eigenvalue* of matrix A and $V(A, \lambda(A))$ is called the *main eigen space* of A.

Theorem 3. *If* $A \in (\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in (\mathbb{R}_\varepsilon^{n\times n})_b$ *and if* A *is not a matrix where all elements are* $[\varepsilon, \varepsilon]$ *and* $V^+(A) \neq \varnothing$, *then* $[\varepsilon, \varepsilon] < \lambda(A)$ *and* $A \overline{\otimes} x = \lambda(A) \overline{\otimes} x, \forall x \in V^+(A)$.

Proof. It is known that matrix A is not a matrix in which each entry belongs to $[\varepsilon, \varepsilon]$ and $V^+(A) \neq \varnothing$. Consequently, $\underline{A}$ and $\overline{A}$ do not have ε as their entries. In addition, since $V^+(A) \neq \varnothing$, there is $x \in V^+(A)$,

$x \approx [\underline{x}, \overline{x}]$ such that $\underline{A} \otimes \underline{x} = \underline{\lambda} \otimes \underline{x}$, $\overline{A} \otimes \overline{x} = \overline{\lambda} \otimes \overline{x}$ or $\underline{x} \in V^+(\underline{A})$, $\overline{x} \in V^+(\overline{A})$ for $\lambda = [\underline{\lambda}, \overline{\lambda}]$. Thus, $V^+(\underline{A}) \neq \varnothing$ and $V^+(\overline{A}) \neq \varnothing$. Therefore, $\underline{\lambda}(\underline{A}) > \varepsilon$, $\overline{\lambda}(\overline{A}) > \varepsilon$ and $\underline{A} \otimes \underline{x} = \underline{\lambda}(\underline{A}) \otimes \underline{x}$, $\overline{A} \otimes \overline{x} = \overline{\lambda}(\overline{A}) \otimes \overline{x}$. Thus, we obtain $\varepsilon = [\varepsilon, \varepsilon] < [\underline{\lambda}(\underline{A}), \overline{\lambda}(\overline{A})] = \lambda(A)$ and $A \overline{\otimes} x = \lambda(A) \overline{\otimes} x$, $\forall x \in V^+(A)$. □

Theorem 4. *If* $A \in I(\mathbb{R})_\varepsilon^{n \times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n \times n})_b$ *and* A *is not a matrix in which each entry belongs to* $[\varepsilon, \varepsilon]$, *then we have*:

(a) $V^+(A) \neq \varnothing$ *if and only if* $\varepsilon < \lambda(A)$ *and* $\forall i \in N, \exists j \in E(A)$ *such that* $j \to i$ *in the* D_A.

(b) *If* $V^+(A) \neq \varnothing$, *then* $V^+(A) = \left\{ \sum_{j \in E(A)}^{\oplus} \alpha_j \overline{\otimes} g_j; \alpha_j \in I(\mathbb{R}) \right\}$, *where* $g_1, g_2, \ldots, g_n$ *are the columns of* $\Gamma(A_\lambda)$.

Proof. Suppose $A \in I(\mathbb{R})_\varepsilon^{n \times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n \times n})_b$ and A is not a matrix in which each entry belongs to $[\varepsilon, \varepsilon]$. Therefore, the lower and upper bound matrices $\underline{A}$ and $\overline{A}$ do not have ε as their entries. From here, we have:

(a) $V^+(\underline{A}) \neq \varnothing \Leftrightarrow \underline{\lambda}(\underline{A}) > \varepsilon$ and $\forall i \in N$, $\exists j \in E(\underline{A})$ such that $j \to i$ in the $D_{\underline{A}}$. Likewise, $V^+(\overline{A}) \neq \varnothing \Leftrightarrow \lambda(\overline{A}) > \varepsilon$ and $\forall i \in N$, $\exists j \in E(\overline{A})$ such that $j \to i$ in the $D_{\overline{A}}$.

(b) If $V^+(\underline{A}) \neq \varnothing$, then $V^+(\underline{A}) = \left\{ \sum_{j \in E(\underline{A})}^{\oplus} \underline{\alpha}_j \otimes \underline{g}_j; \underline{\alpha}_j \in \mathbb{R} \right\}$, where $\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n$ are the columns of $\Gamma(\underline{A}_\lambda)$. Likewise, if $V^+(\overline{A}) \neq \varnothing$, then $V^+(\overline{A}) = \left\{ \sum_{j \in E(\overline{A})}^{\oplus} \overline{\alpha}_j \otimes \overline{g}_j; \overline{\alpha}_j \in \mathbb{R} \right\}$, where $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n$ are the columns of $\Gamma(\overline{A}_\lambda)$.

By (a), we have $V^{+}(A) \neq \emptyset \Leftrightarrow \varepsilon < \lambda(A)$ and $\forall i \in N, \exists j \in E(A)$ such that $j \rightarrow i$ in the D_A. By (b), we can form matrix $\Gamma(A_\lambda)$. If $V^{+}(A) \neq \emptyset$, then $V^{+}(\underline{A}) = \left\{ \sum_{j \in E(A)}^{\oplus} \alpha_j \,\overline{\otimes}\, g_j;\ \alpha_j \in I(\mathbb{R}) \right\}$, where $g_1, g_2, \ldots, g_n$ are the columns of $\Gamma(A_\lambda)$. □

Theorem 4 shows the necessary and sufficient conditions for the existence of finite eigenvectors and how to form the set of finite eigenvectors. The following theorem shows how to establish a set of finite eigenvectors.

Theorem 5. *If* $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_{\varepsilon}^{n \times n})_b$, $\varepsilon < \lambda(A)$, $\Gamma(A_\lambda) = (g_{ij})$ *and* $g_1, g_2, \ldots, g_n$ *are the columns of* $\Gamma(A_\lambda) = (g_{ij})$, *then*:

(a) $i \in E(A)$ *if and only if* $\underline{g}_{ii} = 0$ *and* $\overline{g}_i$ *is an eigenvector matrix* $\overline{A}$.

(b) *If* $i, j \in E(A)$, *then* $i \sim j$ *if and only if* $g_i = \alpha \,\overline{\otimes}\, g_j$ *for* $\alpha \in I(\mathbb{R})$.

Proof. Consider the lower and upper bound matrices $\underline{A}$ and $\overline{A}$. Then $\lambda(\underline{A}) > \varepsilon$, $\lambda(\overline{A}) > \varepsilon$, $\Gamma(\underline{A}_\lambda) = (\underline{g}_{ij})$ and $\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n$ are the columns of $\Gamma(\underline{A}_\lambda)$. Likewise, $\Gamma(\overline{A}_\lambda) = (\overline{g}_{ij})$ and $\overline{g}_1, \overline{g}_2, \ldots, \overline{g}_n$ are the columns of $\Gamma(\overline{A}_\lambda)$. We obtain:

(a) $i \in E(\underline{A}) \Leftrightarrow \underline{g}_{ii} = 0$ and $i \in E(\overline{A}) \Leftrightarrow \overline{g}_{ii} = 0$. Based on the previous results, $i \in E(A) \Leftrightarrow g_i \approx [\underline{g}_i, \overline{g}_i]$ or $g_i \approx [\underline{g}_i, \overline{g}_i^{*}]$. Therefore, $i \in E(A) \Leftrightarrow \underline{g}_{ii} = 0$ and the upper bound vector is the eigenvector of the matrix $\overline{A}$.

(b) If $i, j \in E(\underline{A})$, then $\underline{g}_i = \underline{\alpha} \otimes \underline{g}_j$ for $\underline{\alpha} \in I(\mathbb{R})$ if and only if $i \sim j$. Likewise, if $i, j \in E(\overline{A})$, then $\overline{g}_i = \overline{\alpha} \otimes \overline{g}_j$ or $\overline{g}_i^{*} = \overline{\alpha} \otimes \overline{g}_j^{*}$ for

$\overline{\alpha} \in I(\mathbb{R})$ if and only if $i \sim j$. As a result, if $i, j \in E(A)$, then $g_i = \alpha \,\overline{\otimes}\, g_j$ for $\alpha \in I(\mathbb{R})$ if and only if $i \sim j$. □

Corollary 1. *Suppose $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$. If $\varepsilon < \lambda(A)$, $g_1, g_2, \ldots, g_n$ are the columns of $\Gamma(A_\lambda)$ and $V^+(A) \neq \varnothing$, then*

$$V^+(A) = \left\{ \sum\nolimits_{j\in E^*(A)}^{\overline{\oplus}} \alpha_j \,\overline{\otimes}\, g_j;\ \alpha_j \in I(\mathbb{R}) \right\},$$

where $E^*(A)$ is a maximal set of critical points of A which are not equivalent.

The following theorem presents a set of eigenvectors for an arbitrary irreducible matrix.

Theorem 6. *Every irreducible matrix $A \in I(\mathbb{R})_{\varepsilon}^{n\times n}$ $(n > 1)$ has a unique eigenvalue $\lambda(A)$ and the set of eigenvectors is $V(A) - \{\varepsilon\} = V^+(A) = \left\{ \sum\nolimits_{j\in E^*(A)}^{\overline{\oplus}} \alpha_j \,\overline{\otimes}\, g_j \,|\, \alpha_j \in I(\mathbb{R}) \right\}$, where $g_1, g_2, \ldots, g_n$ are the columns of $\Gamma(A_\lambda)$.*

Proof. Note that the lower and upper bound matrices $\underline{A}$ and $\overline{A}$ are irreducible matrices. By Theorem 5, matrices $\underline{A}$ and $\overline{A}$ have unique eigenvalues $\underline{\lambda}(\underline{A})$ and $\overline{\lambda}(\overline{A})$, respectively,

$$V(\underline{A}) - \{\varepsilon\} = V^+(\underline{A}) = \left\{ \sum\nolimits_{j\in E^*(\underline{A})}^{\oplus} \underline{\alpha}_j \otimes \underline{g}_j;\ \underline{\alpha}_j \in \mathbb{R} \right\},$$

where $\underline{g}_1, \underline{g}_2, \ldots, \underline{g}_n$ are the columns of $\Gamma(\underline{A}_\lambda)$, and $E^*(\underline{A})$ is an arbitrary set of maximum critical points of $\underline{A}$ which are not equivalent. Likewise,

$$V(\overline{A}) - \{\varepsilon\} = V^+(\overline{A}) = \left\{ \sum\nolimits_{j\in E^*(\overline{A})}^{\oplus} \overline{\alpha}_j \otimes \overline{g}_j;\ \overline{\alpha}_j \in \mathbb{R} \right\},$$

where $\overline{g}_1, \overline{g}_2, ..., \overline{g}_n$ are the columns of $\Gamma(\overline{A}_\lambda)$, and $E^*(\overline{A})$ is an arbitrary maximum set of critical points of $\overline{A}$ which are not equivalent. Therefore, $V(A) - \{\varepsilon\} = V^+(A) = \left\{ \sum_{j \in E^*(A)}^{\oplus} \alpha_j \overline{\otimes} g_j ; \alpha_j \in I(\mathbb{R}) \right\}$ with $\alpha_j \overline{\otimes} g_j \approx [\underline{\alpha}_j \otimes \underline{g}_j, \overline{\alpha}_j \otimes \overline{g}_j]$. □

Eigenvalues and eigenvectors of a reducible matrix

Next, we will discuss eigenvalues and eigenvectors for any reducible matrix. A reducible matrix possibly has more than one eigenvalue.

Definition 3. Let $N = \{1, 2, ..., n\}$, $K = \{i_1, i_2, ..., i_k\} \subseteq N$ with $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $A[K] \approx [\underline{A}[k], \overline{A}[k]]$ be the main interval submatrix of the matrix $A = [A_{ij}] \in I(\mathbb{R})_\varepsilon^{n \times n}$, $A \approx [\underline{A}, \overline{A}]$, i.e., $\begin{bmatrix} A_{i_1 i_1} & \cdots & A_{i_1 i_k} \\ \cdots & \cdots & \cdots \\ A_{i_k i_1} & \cdots & A_{i_k i_k} \end{bmatrix}$ and $x[K] \approx [\underline{x}[K], \overline{x}[K]]$ states the subvector $(x_{i_1}, x_{i_2}, ..., x_{i_k})^T$ of the vector $(x_1, x_2, ..., x_n)^T \in I(\mathbb{R})_\varepsilon^n$. Furthermore, if $D = (N, E)$ is a directed graph and $K \subseteq N$, then by $D[K]$ we mean a directed subgraph induced by D, i.e., $D[K] = (K, E \cap (K \times K))$, so $D_{A[K]} = D[K]$.

Definition 4. Let A, $B \in I(\mathbb{R})_\varepsilon^{n \times n}$. A symbol $A \sim B$ means that A can be obtained from B by permutating the columns and rows.

The following lemma describes the eigenvalues and eigenvectors of two equivalent matrices.

Lemma 3. *If $A \sim B$, then $\Lambda(A) = \Lambda(B)$ and there is a bijective function between $V(A)$ and $V(B)$.*

Proof. Let $A, B \in I(\mathbb{R})_\varepsilon^{n \times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n \times n})_b$ and $B \approx [\underline{B}, \overline{B}] \in I(\mathbb{R}_\varepsilon^{n \times n})_b$. Since $A \sim B$, $\underline{A} \sim \underline{B}$ and $\overline{A} \sim \overline{B}$. Thus, we have $\Lambda(\underline{A}) = \Lambda(\underline{B})$

and there is a bijective function between $V(\underline{A})$ and $V(\underline{B})$, as well as $\Lambda(\overline{A}) = \Lambda(\overline{B})$ and there is a bijective function between $V(\overline{A})$ and $V(\overline{B})$. Therefore, $\Lambda(A) = \Lambda(B)$ and there is a bijective function between $V(A)$ and $V(B)$. □

Lemma 4. *Suppose* $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $\lambda \in \Lambda(A)$ *and* $x \in V(A, \lambda)$. *If* $x \notin V^+(A, \lambda)$, *then* $n > 1$, $A \sim \begin{pmatrix} A^{(11)} & A^{(21)} \\ \varepsilon & A^{(22)} \end{pmatrix}$, $\lambda = \lambda(A^{(22)})$ *and* A *is a reducible matrix.*

Proof. Suppose $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$, $\lambda = [\underline{\lambda}, \overline{\lambda}] \in \Lambda(A)$ and $x = [x_1, x_2, ..., x_n]^T = [[\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], ..., [\underline{x}_n, \overline{x}_n]]^T \in V(A, \lambda)$. Therefore, $\underline{\lambda} \in \Lambda(\underline{A})$, $\overline{\lambda} \in \Lambda(\overline{A})$, $\underline{x} = [\underline{x}_1, \underline{x}_2, ..., \underline{x}_n] \in V(\underline{A}, \underline{\lambda})$ and $\overline{x} = [\overline{x}_1, \overline{x}_2, ..., \overline{x}_n] \in V(\overline{A}, \overline{\lambda})$. If $\underline{x} \notin V^+(\underline{A}, \underline{\lambda})$, then $n > 1$, and $\underline{A} \sim \begin{bmatrix} \underline{A}^{(11)} & \underline{A}^{(21)} \\ \varepsilon & \underline{A}^{(22)} \end{bmatrix}$, $\underline{\lambda} = \lambda(\underline{A}^{(22)})$ and $\underline{A}$ is a reducible matrix. Likewise, if $\overline{x} \notin V^+(\overline{A}, \overline{\lambda})$, then $n > 1$, $\overline{A} \sim \begin{bmatrix} \overline{A}^{(11)} & \overline{A}^{(21)} \\ \varepsilon & \overline{A}^{(22)} \end{bmatrix}$, $\overline{\lambda} = \lambda(\overline{A}^{(22)})$ and $\overline{A}$ is a reducible matrix. Since $x \notin V^+(A, \lambda)$, we have $n > 1$, and $A \sim \begin{bmatrix} A^{(11)} & A^{(21)} \\ \varepsilon & A^{(22)} \end{bmatrix}$, $\lambda = \lambda(A^{(22)})$ and A is a reducible matrix. □

The necessary and sufficient conditions for a matrix to be irreducible are presented in the following theorem.

Theorem 7. *Given a matrix* $A \in I(\mathbb{R})_{\max}^{n\times n}$, *then* $V(A) = V^+(A)$ *if and only if* A *is an irreducible matrix.*

Proof. Suppose $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$. Hence, $V(\underline{A}) = V^+(\underline{A})$ if and only if $\underline{A}$ is an irreducible matrix. Likewise, $V(\overline{A}) = V^+(\overline{A})$

if and only if $\overline{A}$ is an irreducible matrix. By Definition 2, we obtain $V(A) = V^+(A)$ if and only if A is an irreducible matrix. □

Every reducible matrix $A = (A_{ij}) \in I(\mathbb{R})_\varepsilon^{n\times n}$ can be transformed by permutating rows and columns of matrix A into a Frobenius normal form (FNF), namely $\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{r1} \\ \varepsilon & A_{22} & \cdots & A_{r2} \\ \cdots & \cdots & \cdots & \cdots \\ \varepsilon & \varepsilon & \cdots & A_{rr} \end{bmatrix}$, where $A_{11}, ..., A_{rr}$ is an irreducible square submatrix of A.

Definition 5. Given an FNF matrix A. A simplification of any directed graph is a directed graph $C_A = (\{N_1, ..., N_r\}, \{(N_i, N_j) | \exists k \in N_i, \exists l \in N_j\})$ such that $A_{lk} > [\varepsilon, \varepsilon]$.

If there is a path from a point in N_i to a point inside the N_j in D_A, then we denote $N_i \to N_j$.

Lemma 5. *If* $x \in V(A)$, $N_i \to N_j$ *and* $x[N_j] \neq \varepsilon$, *then* $x[N_i]$ *is finite, particularly,* $x[N_j]$ *is finite.*

Proof. Let

$$x \in V(A),\ x = [x_1, x_2, ..., x_n]^T = [[\underline{x}_1, \overline{x}_1], [\underline{x}_2, \overline{x}_2], ..., [\underline{x}_n, \overline{x}_n]]^T.$$

Then $x \approx [\underline{x}, \overline{x}]$ with $\underline{x} = [\underline{x}_1, \underline{x}_2, ..., \underline{x}_n]^T \in V(\underline{A})$ and $\overline{x} = [\overline{x}_1, \overline{x}_2, ..., \overline{x}_n]^T \in V(\overline{A})$. Suppose $N_i \to N_j$ and $\underline{x}[N_j] \neq \varepsilon$. Then $\underline{\mathbf{x}}[N_i]$ finite, particularly, $\underline{x}[N_j]$ is finite. Similarly, since $\overline{x}[N_i] \neq \varepsilon$, we have $\overline{x}[N_i]$ is finite, particularly, $\overline{x}[N_j]$ is finite. Consequently, $x[N_i]$ is finite, particularly, $\mathbf{x}[N_j]$ is finite. □

Theorem 8 (Spectral theorem). *If* $A \in I(\mathbb{R})_\varepsilon^{n\times n}$ *is an FNF matrix, then* $\Lambda(A) \approx \{\lambda(A_{jj}) | \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})\}$.

Proof. Suppose $A \in I(\mathbb{R})_\varepsilon^{n\times n}$ and $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$. Since A is an *FNF* matrix, $\underline{A}$ and $\overline{A}$ are *FNF* matrices. Hence, $\Lambda(\underline{A}) \approx \{\underline{\lambda}(A_{jj}) | \lambda(\underline{A}_{jj}) = \max_{N_i \to N_j} \lambda(\underline{A}_{ii})\}$ and

$$\Lambda(\overline{A}) \approx \{\overline{\lambda}(\overline{A}_{jj}) | \overline{\lambda}(\overline{A}_{jj}) = \max_{N_i \to N_j} \overline{\lambda}(\overline{A}_{ii})\}.$$

We obtain $\Lambda(A) \approx \{\lambda(A_{jj}) | \lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})\}$, with $\lambda(A_{jj}) = [\underline{\lambda}(\underline{A}_{jj}), \overline{\lambda}(\overline{A}_{jj})]$. □

Here is the definition of spectral and consequences of Theorem 8.

Definition 6. Given $A \in I(\mathbb{R})_\varepsilon^{n\times n}$ which is an *FNF matrix*. If $\lambda(A_{jj}) = \max_{N_i \to N_j} \lambda(A_{ii})$, then A_{jj} (and also N_j or simply *j*) is called as *spectral*.

Corollary 2. *All initial classes of* C_A *are spectral*.

Corollary 3. *The number of eigenvalues is* $|\Lambda(A)|$, $1 \le |\Lambda(A)| \le n$ *for every* $A \in I(\mathbb{R})_\varepsilon^{n\times n}$.

Corollary 4. *The set* $V(A) = V(A, \lambda(A))$ *if and only if all the initial classes have the same eigenvalue* $\lambda(A)$.

The following discussion is about how to determine all the sets of eigenvectors of a matrix.

Definition 7. Given $A \in I(\mathbb{R})_\varepsilon^{n\times n}$ which is an *FNF* matrix, $N_1, N_2, \ldots, N_r$ are the classes of A and $R = \{1, 2, \ldots, r\}$. Suppose that $\lambda \in \Lambda(A)$ and $\lambda > \varepsilon$, we denote $I(\lambda) = \{i \in R | \lambda(N_i) = \lambda,\ N_i \text{ spectral}\}$ and $E(\lambda) = \bigcup_{i \in I(\lambda)} E(A_{ii}) = \{j \in N | \underline{g}_{jj} = 0,\ j \in \bigcup_{i \in I(\lambda)} N_i\}$ with $\Gamma(\lambda^{-1} \overline{\otimes} A) = [g_{ij}]$.

Definition 8. Given $i, j \in E(\lambda)$. Points i and j are said to be λ-*equivalent* if and only if i and j are included in the same cycle, with the average cycle λ, and denoted by $i \sim_\lambda j$.

Theorem 9. *If* $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$ *and* $\lambda \in I\Lambda(A)$, $[\varepsilon, \varepsilon] < \lambda$, *then* $g_k \in I(\mathbb{R})_\varepsilon^n$ *for every* $k \in E(\lambda)$ *and* $V(A, \lambda)$ *is a linear combination of* g_k*'s, where a* g_k *is taken from an equivalent class* $\sim_\lambda$.

Proof. It is known that $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$, $\lambda = [\underline{\lambda}, \overline{\lambda}] \in \Lambda(A)$ and $[\varepsilon, \varepsilon] < [\underline{\lambda}, \overline{\lambda}]$. Then the columns of the matrix $\Gamma(A_\lambda)$ with lower bound and upper bound of the main diagonal elements are 0 and eigenvectors of A are eigenvectors corresponding to λ. The set $V(A, \lambda)$ is a linear combination of g_k's, where a g_k is taken one from each equivalent class in the $(E(\lambda), \sim)$. □

Furthermore, an equivalent class in $(E(\lambda), \sim)$ is called as the *equivalent class* $\sim_\lambda$.

Corollary 5. *If* $A \in I(\mathbb{R})_\varepsilon^{n\times n}$ *and* $\lambda \in \Lambda(A)$, $[\varepsilon, \varepsilon] < [\underline{\lambda}, \overline{\lambda}]$, *then* $V(A, \lambda) = \{\Gamma(\lambda^{-1} \overline{\otimes} A) \overline{\otimes} z \mid z \in I(\mathbb{R})_\varepsilon^n,\ z_j = [\varepsilon, \varepsilon]$ *for all* $j \notin E(\lambda)\}$.

Theorem 10. *The set* $V^+(A) = \varnothing$ *if and only if* $\lambda(A)$ *is an eigenvalue for all end classes.*

Proof. It is known that $A \in I(\mathbb{R})_\varepsilon^{n\times n}$, $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R}_\varepsilon^{n\times n})_b$, $\lambda(A) = [\underline{\lambda}, \overline{\lambda}]$ and $V^+(A, \lambda) \approx V^+([\underline{A}, \overline{A}], [\underline{\lambda}, \overline{\lambda}])$. From here, the set $V^+(\underline{A}) \neq \varnothing$ if and only if $\underline{\lambda}(\underline{A})$ is an eigenvalue for all end classes. Likewise, the set $V^+(\overline{A}) \neq \varnothing$ if and only if $\overline{\lambda}(\overline{A})$ is an eigenvalue for all end classes. As a result, the set $V^+(A) = \varnothing$ if and only if $\lambda(A)$ is an eigenvalue for all end classes. □

Corollary 6. *The set* $V^{+}(A) = \varnothing$ *if and only if an end class has eigenvalues less than* $\lambda(A)$.

Theorem 11. *Let* $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$ *be the FNF matrix and* $N_1, N_2, \ldots, N_r$ *be classes of* A *with* $R = \{1, 2, \ldots, r\}$. *If* $K = \{1, 2, \ldots, k\}$ *and* $\Lambda(A) = \{\lambda_i \,|\, i \in K\}$, *then for* $k \leq r$, *there are* $\Gamma^{(1)}, \Gamma^{(2)}, \ldots, \Gamma^{(k)}$ *such that* $\forall i \in K,\ Im(\Gamma^{(i)}) = V(A, \lambda_i)$.

Proof. Since the number of eigenvalues does not exceed the number of classes, $k \leq r$. If $\lambda_i \in \Lambda(A)$, then $V(A, \lambda_i) = \{\Gamma(\lambda_i^{-1} \overline{\otimes} A) \overline{\otimes} z \,|\, z \in I(\mathbb{R})_{\varepsilon}^{n}, z_j = [\varepsilon, \varepsilon]$ for all $j \notin IE(\lambda_i)\}$. Suppose that $E(\lambda_i) = \{e_1, e_2, \ldots, e_l\}$. Then $\Gamma^{(i)} = (g_{ij})$ and $g_1, g_2, \ldots, g_l$ are the columns of $\Gamma^{(i)}$, where g_i is equal to the e_i th column of $\Gamma(\lambda_i^{-1} \overline{\otimes} A)$ so that $Im(\Gamma^{(i)}) = V(A, \lambda_i)$. □

Sub-eigenvector

Below is given a definition of sub-eigenvector.

Definition 9. Given a matrix $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$. Vector $x \in I(\mathbb{R})_{\varepsilon}^{n}$, $x \neq (\varepsilon, \varepsilon, \ldots, \varepsilon)$ and $\lambda \in I(\mathbb{R})_{\varepsilon}$ satisfying

$$A \overline{\otimes} x \leq \lambda \overline{\otimes} x,$$

are, respectively, called *sub-eigenvectors* and *eigenvalues* of matrix A.

We denote

$$V^{*}(A, \lambda) = \{x \in I(\mathbb{R})^{n} \,|\, x \approx [\underline{x}, \overline{x}] \ni \underline{x} \in V^{*}(\underline{A}, \underline{\lambda});\ \overline{x} \in V^{*}(\overline{A}, \overline{\lambda})\},$$

$$V^{*}(A) = V^{*}(A, \lambda(A)),$$

$$V_0^{*}(A) = V^{*}(A, 0),$$

where for $A \neq \varepsilon \in \mathbb{R}_{\varepsilon}^{n \times n}$, $V^{*}(A, \lambda) = \{x \in \mathbb{R}^{n} \,|\, A \otimes x \leq \lambda \otimes x\}$.

Here, we give criteria for finite sub-eigenvector.

Theorem 12. *Given a matrix* $A \neq \varepsilon \in I(\mathbb{R})_{\varepsilon}^{n \times n}$. *Inequality* $A \overline{\otimes} x \leq \lambda \overline{\otimes} x$ *has a finite solution if and only if* $\lambda \geq \lambda(A)$ *and* $\lambda > \varepsilon$.

Proof. Inequality $A \overline{\otimes} x \leq \lambda \overline{\otimes} x$ has a finite solution if and only if $x \in V^{*}(A, \lambda)$. From the definition $V^{*}(A, \lambda)$, then $x \approx [\underline{x}, \overline{x}]$ such that $\underline{x} \in V^{*}(\underline{A}, \underline{\lambda})$; $\overline{x} \in V^{*}(\overline{A}, \overline{\lambda})$. Since $\underline{x} \in V^{*}(\underline{A}, \underline{\lambda})$ if and only if $\underline{\lambda} \geq \underline{\lambda}(\underline{A})$ and $\underline{\lambda} < \varepsilon$ and $\overline{x} \in V^{*}(\overline{A}, \overline{\lambda})$ if and only if $\overline{\lambda} \geq \overline{\lambda}(\overline{A})$ and $\overline{\lambda} > \varepsilon$, we have $\lambda \geq \lambda(A)$ and $\lambda > \varepsilon$. □

The above theorem gives the necessary and sufficient conditions for the existence of finite sub-eigenvectors and eigenvalues of a matrix. Furthermore, we give a description of the set of finite sub-eigenvectors.

Theorem 13. *Given a matrix* $A \neq \varepsilon \in I(\mathbb{R})_{\varepsilon}^{n \times n}$. *If* $\lambda \geq \lambda(A)$ *and* $\lambda > \varepsilon$, *then the set of finite sub-eigenvectors* $V^{*}(A, \lambda) = \{\Delta(\lambda^{-1} \overline{\otimes} A) \overline{\otimes} u \mid u \in I(\mathbb{R})_{\varepsilon}^{n}\}$.

Proof. If $\lambda \geq \lambda(A)$ and $\lambda > \varepsilon$, then $A \overline{\otimes} x \leq \lambda \overline{\otimes} x$ has a finite solution. From the definition $V^{*}(A, \lambda)$, then $x \approx [\underline{x}, \overline{x}]$ such that $\underline{x} \in V^{*}(\underline{A}, \underline{\lambda})$; $\overline{x} \in V^{*}(\overline{A}, \overline{\lambda})$. Since $\underline{x} \in V^{*}(\underline{A}, \underline{\lambda})$ if and only if $\underline{\lambda} \geq \underline{\lambda}(\underline{A})$ and $\underline{\lambda} > \varepsilon$ and $\overline{x} \in V^{*}(\overline{A}, \overline{\lambda})$ if and only if $\overline{\lambda} \geq \overline{\lambda}(\overline{A})$ and $\overline{\lambda} > \varepsilon$, $V^{*}(\underline{A}, \underline{\lambda}) = \{\Delta(\underline{\lambda}^{-1} \overline{\otimes} \underline{A}) \overline{\otimes} u \mid u \in I(\mathbb{R})_{\varepsilon}^{n}\}$ and $V^{*}(\overline{A}, \overline{\lambda}) = \{\Delta(\overline{\lambda}^{-1} \overline{\otimes} \overline{A}) \overline{\otimes} u \mid u \in I(\mathbb{R})_{\varepsilon}^{n}\}$, so $V^{*}(A, \lambda) = \{\Delta(\lambda^{-1} \overline{\otimes} A) \overline{\otimes} u \mid u \in I(\mathbb{R})_{\varepsilon}^{n}\}$. □

By Theorem 13, if $x \in I(\mathbb{R})_{\varepsilon}^{n}$, then we obtain a description of the set of sub-eigenvectors on the following theorem:

Theorem 14. *Given a matrix* $A \neq \varepsilon \in I(\mathbb{R})_{\varepsilon}^{n \times u}$. *If* $\lambda \geq \lambda(A)$ *and* $\lambda > \varepsilon$, *then*

$$A \overline{\otimes} x \leq \lambda \overline{\otimes} x, \; x \in I(\mathbb{R})_{\varepsilon}^{n}$$

if and only if

$$x = \Delta(\lambda^{-1} \overline{\otimes} A) \overline{\otimes} u,\ u \in I(\mathbb{R})^n_\varepsilon.$$

Furthermore, we give the properties of the sub-eigenvectors in the following lemmas.

Lemma 6. *Given an irreducible matrix* $A \neq \varepsilon \in I(\mathbb{R})^{n\times n}_\varepsilon$, $A \overline{\otimes} x \leq \lambda \overline{\otimes} x$ *with* $x \neq \varepsilon$, $\lambda \in I(\mathbb{R})_\varepsilon$, *then* $x \in I(\mathbb{R})^n$.

Proof. If $A \overline{\otimes} x \leq \lambda \overline{\otimes} x$ with $x \neq \varepsilon$, $\lambda \in I(\mathbb{R})_\varepsilon$, then $x \approx [\underline{x}, \overline{x}]$ such that $\underline{A} \overline{\otimes} \underline{x} \leq \underline{\lambda} \overline{\otimes} \underline{x}$ with $\underline{x} \neq \varepsilon$, $\underline{\lambda} \in I(\mathbb{R})_\varepsilon$ and $\overline{A} \overline{\otimes} \overline{x} \leq \overline{\lambda} \overline{\otimes} \overline{x}$ with $\overline{x} \neq \varepsilon$, $\overline{\lambda} \in I(\mathbb{R})_\varepsilon$. Consequently, $\underline{x} \in I(\mathbb{R})^n$ and $\overline{x} \in I(\mathbb{R})^n$, so $x \in I(\mathbb{R})^n$. □

Lemma 7. *Given a matrix* $A \in I(\mathbb{R})^{n\times n}_\varepsilon$ *and* $\lambda(A) > \varepsilon$. *If* $x \in V^*(A)$ *and* $(i, j) \in E_C(A)$, *then* $A_{ij} \overline{\otimes} x_j = \lambda(A) \overline{\otimes} x_i$.

Proof. Since $x \in V^*(A)$ and $V^*(A) = V^*(A, \lambda(A))$, $x \in I(\mathbb{R})^n$ and $A \overline{\otimes} x \leq \lambda(A) \overline{\otimes} x$, so $x \approx [\underline{x}, \overline{x}]$ such that $\underline{x}, \overline{x} \in I(\mathbb{R})^n$ and $\underline{A} \overline{\otimes} \underline{x} \leq \underline{\lambda}(\underline{A}) \overline{\otimes} \underline{x}$, $\overline{A} \overline{\otimes} \overline{x} \leq \overline{\lambda}(\overline{A}) \overline{\otimes} \overline{x}$. Consequently, $\underline{x} \in V^*(\underline{A}, \underline{\lambda}(\underline{A}))$ and $\overline{x} \in V^*(\overline{A}, \overline{\lambda}(\overline{A}))$. Since $V^*(\underline{A}) = V^*(\underline{A}, \underline{\lambda}(\underline{A}))$ and $V^*(\overline{A}) = V^*(\overline{A}, \overline{\lambda}(\overline{A}))$, $\underline{x} \in V^*(\underline{A})$ and $\overline{x} \in V^*(\overline{A})$. By $(i, j) \in E_C(A)$, then $(i, j) \in E_C(\underline{A})$ and $(i, j) \in E_C(\overline{A})$. Therefore, we have $\underline{A}_{ij} \overline{\otimes} \underline{x}_j = \underline{\lambda}(\underline{A}) \overline{\otimes} \underline{x}_i$ and $\overline{A}_{ij} \overline{\otimes} \overline{x}_j = \overline{\lambda}(\overline{A}) \overline{\otimes} \overline{x}_i$, so $A_{ij} \overline{\otimes} x_j = \lambda(A) \overline{\otimes} x_i$. □

Lemma 8. *Given a matrix* $A \in I(\mathbb{R})^{n\times n}_\varepsilon$ *and* $\lambda(A) > \varepsilon$. *If* $z \in V^*(A)$, *then* $\lambda(A) = z^* \overline{\otimes} A \overline{\otimes} z = \min_{x \in I(\mathbb{R})^n} x^* \overline{\otimes} A \overline{\otimes} x$, *where* $z^* = -z^T$ *is the conjugate transpose of* z.

Proof. Since $z \in V^*(A)$ and $V^*(A) = V^*(A, \lambda(A))$, $z \in I(\mathbb{R})^n$ and $A \overline{\otimes} z \leq \lambda(A) \overline{\otimes} z$, so $z \approx [\underline{z}, \overline{z}]$ such that $\underline{z}, \overline{z} \in I(\mathbb{R})^n$ and $\underline{A} \overline{\otimes} \underline{z} \leq \underline{\lambda}(\underline{A}) \overline{\otimes} \underline{z}$, $\overline{A} \otimes \overline{x} \leq \overline{\lambda}(\overline{A}) \overline{\otimes} \overline{x}$. Consequently, $\underline{z} \in V^*(\underline{A}, \underline{\lambda}(\underline{A}))$ and $\overline{z} \in V^*(\overline{A}, \overline{\lambda}(\overline{A}))$. Since $V^*(\underline{A}) = V^*(\underline{A}, \underline{\lambda}(\underline{A}))$ and $V^*(\overline{A}) = V^*(\overline{A}, \overline{\lambda}(\overline{A}))$, $\underline{z} \in V^*(\underline{A})$ and $\overline{z} \in V^*(\overline{A})$. Hence, we have

$$\underline{\lambda}(\underline{A}) = \underline{z}^* \overline{\otimes} \underline{A} \overline{\otimes} \underline{z} = \min_{\underline{x} \in I(\mathbb{R})^n} \underline{x}^* \overline{\otimes} \underline{A} \overline{\otimes} \underline{x}$$

and

$$\overline{\lambda}(\overline{A}) = \overline{z}^* \overline{\otimes} \overline{A} \overline{\otimes} \overline{z} = \min_{\overline{x} \in I(\mathbb{R})^n} \overline{x}^* \overline{\otimes} \overline{A} \overline{\otimes} \overline{x}.$$

And thus

$$\lambda(A) = z^* \overline{\otimes} A \overline{\otimes} z = \min_{x \in I(\mathbb{R})^n} x^* \overline{\otimes} A \overline{\otimes} x.$$ □

We give the connection between the average maximum cycle with sub-eigenvector as follows.

Lemma 9. *If* $A \in I(\mathbb{R})_\varepsilon^{n \times n}$, *then*

$$\lambda(A) = \inf\{\lambda \mid A \overline{\otimes} x \leq \lambda \overline{\otimes} x, x \in I(\mathbb{R})^n\}.$$

If $\lambda(A) < \varepsilon$ *or* $A = \varepsilon$, *then the infimum is reached.*

Proof. If $A \approx [\underline{A}, \overline{A}] \in I(\mathbb{R})_\varepsilon^{n \times n}$, then

$$\underline{\lambda}(\underline{A}) = \inf\{\underline{\lambda} \mid \underline{A} \overline{\otimes} \underline{x} \leq \underline{\lambda} \overline{\otimes} \underline{x}, \underline{x} \in I(\mathbb{R})^n\}$$

and $\overline{\lambda}(\overline{A}) = \inf\{\overline{\lambda} \mid \overline{A} \otimes \overline{x} \leq \overline{\lambda} \overline{\otimes} \overline{x}, \overline{x} \in I(\mathbb{R})^n\}$ and so $\lambda(A) = \inf\{\lambda \mid A \overline{\otimes} x \leq \lambda \overline{\otimes} x, x \in I(\mathbb{R})^n\}$. □

3. Concluding Remarks

Based on the above discussion, some conclusions can be drawn as follows:

(1) The maximum average cycle is an eigenvalue of each square matrix on max-plus algebra interval, and the average is the only eigenvalue corresponding to the finite eigenvector. For an irreducible square matrix, the eigenvalues are unique, namely the average maximum cycle, and the corresponding eigenvectors are finite eigenvectors.

(2) Criterion for the existence of maximum of a finite eigenvector for the matrix A is the following: for a given matrix $A \in I(\mathbb{R})_{\varepsilon}^{n \times n}$, the finite eigenvector of the matrix A exists:

(a) If the maximum of columns j of the matrix $\Gamma(A_{\lambda})$ is in $I(\mathbb{R})^{n}$, where j is at a critical point set of A.

(b) If the average maximum cycle of the matrix A is finite and in digraphs D_A, for each point i digraph D_A, there is a critical point j so that the point j can be achieved by the point i.

(c) If the value of $\lambda(A)$ is an eigenvalue of all final classes in each super block.

(d) When a final grade has eigenvalues less than $\lambda(A)$.

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