[Mathematics] Manuscript ID: mathematics-2967439 - Article Processing Charge Confirmation

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on behalf of
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Mon 4/1/2024 4:53 PM

To:José Luís Da Silva <joses@staff.uma.pt>
Cc:Herry P Suryawan <herrypribs@usd.ac.id>;Mathematics Editorial Office <mathematics@mdpi.com>

Dear Professor da Silva,

Thank you very much for submitting your manuscript to Mathematics:

Journal name: Mathematics
Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024
E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
Probability and Statistics
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[Mathematics] Manuscript ID: mathematics-2967439 - Co-Authorship Confirmation

Mathematics Editorial Office <mathematics@mdpi.com>
Sun 3/31/2024 7:49 PM
To:Herry P Suryawan <herrypribs@usd.ac.id>
Cc:Mathematics Editorial Office <mathematics@mdpi.com>

Dear Professor Suryawan,

We are writing to let you know that we have received the below submission to Mathematics for which you are listed as a co-author.

Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024

In order to confirm your connection to this submission, please click here to confirm your co-authorship:
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Kind regards,
Mathematics Editorial Office

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José Luís da Silva <joses@staff.uma.pt>

Mon 4/1/2024 5:45 PM

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Reply-To: info@preprints.org

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Mathematics Editorial Office

Re: MSC Needed: [Mathematics] Manuscript ID: mathematics-2967439 - Assistant Editor Assigned

José Luís da Silva <joses@staff.uma.pt>

Wed 4/3/2024 3:18 PM

To:krystal.wang@mdpi.com <krystal.wang@mdpi.com>
Cc:mathematics@mdpi.com <mathematics@mdpi.com>;Herry P Suryawan <herrypribs@usd.ac.id>

Dear Ms. Krystal Wang,

here I am sending you the MSC2020 code for the manuscript:

37P30, 60G22; 60G10; 28C05

Thanks in advance
José Luís

> On 3 Apr 2024, at 2:51 AM, Krystal Wang / MDPI <krystal.wang@mdpi.com> wrote:
>
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>
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> provide the MSC code of this manuscript.
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> On 2024/4/1 17:54, Krystal Wang wrote:
>> Dear Professor da Silva,
>> Your paper has been assigned to Krystal Wang, who will be your main point of contact as your paper is processed further.
>> Journal: Mathematics Manuscript ID: mathematics-2967439 Title: Green Measures for a Class of non-Markov Processes Authors: Herry P. Suryawan, José L. da Silva * Received: 31 Mar 2024 E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
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[Mathematics] Manuscript ID: mathematics-2967439 - Major Revisions

krystal.wang@mdpi.com <krystal.wang@mdpi.com>
on behalf of
Mathematics Editorial Office <mathematics@mdpi.com>

Thu 4/11/2024 9:56 AM

To:José Luís Da Silva <joses@staff.uma.pt>
Cc:Herry P Suryawan <herrypribs@usd.ac.id>;Mathematics Editorial Office <mathematics@mdpi.com>

Dear Professor da Silva,

Thank you again for your manuscript submission. Please be aware that there may be an additional report to come. We will contact you when the final report is received.

Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024
E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
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Kind regards,
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[Mathematics] Manuscript ID: mathematics-2967439 - Revision Reminder

krystal.wang@mdpi.com <krystal.wang@mdpi.com>
on behalf of
Mathematics Editorial Office <mathematics@mdpi.com>
Thu 4/18/2024 12:52 PM

To:José Luís Da Silva <joses@staff.uma.pt>
Cc:Herry P Suryawan <herrypribs@usd.ac.id>;Mathematics Editorial Office <mathematics@mdpi.com>

Dear Professor da Silva,

We sent a revision request for the following manuscript on 11 April 2024.

Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024
E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
Probability and Statistics
https://www.mdpi.com/journal/mathematics/sections/probability_and_statistics_theory

Please could you update us on the progress of your revisions?

If you have finished your revisions, please upload the revised version together with your responses to the reviewers. The date that we hope to receive your revised manuscript is 21 April 2024.

You can find your manuscript and review reports at this link:
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If you have any questions, please feel free to contact us.

Thank you in advance for your kind cooperation and we look forward to hearing from you soon.

Kind regards,
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Self-Cited Papers Revision: [Mathematics] Manuscript ID: mathematics-2967439 - Revised Version Received

Krystal Wang / MDPI <krystal.wang@mdpi.com>
Wed 4/24/2024 9:03 AM
To:José Luís Da Silva <joses@staff.uma.pt>
Cc:mathematics@mdpi.com <mathematics@mdpi.com>;Herry P Suryawan <herrypribs@usd.ac.id>

Dear Dr. da Silva,

Glad to contact you. We have received your revised version and found that there are 5 your own papers (Ref 3,4,14,16,17) cited in the new version.

Due to the high-self citation, we sincerely suggest removing a few of the author's own articles in References or citing other authors' papers instead during revision.Please send your revised version via email.

Thank you very much for your understanding and cooperation. In case of any questions, please feel free to contact us.

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Ms. Krystal Wang
Section Managing Editor
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On 2024/4/24 9:52, Mathematics Editorial Office wrote:
> Dear Professor da Silva,
>
> Thank you very much for providing the revised version of your paper:
>
> Manuscript ID: mathematics-2967439 Type of manuscript: Article Title:
> Green Measures for a Class of non-Markov Processes Authors: Herry P.
> Suryawan, José L. da Silva * Received: 31 Mar 2024 E-mails:
> herrypribs@usd.ac.id, joses@staff.uma.pt Probability and Statistics
> https://www.mdpi.com/journal/mathematics/sections/probability_and_statistics_theory

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> We will continue processing your paper and will keep you informed
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[Mathematics] Manuscript ID: mathematics-2967439 - Accepted for Publication

leif.qi@mdpi.com <leif.qi@mdpi.com>
on behalf of
Mathematics Editorial Office <mathematics@mdpi.com>

Thu 4/25/2024 11:09 AM

To:José Luís Da Silva <joses@staff.uma.pt>
Cc:Herry P Suryawan <herrypribs@usd.ac.id>;Mathematics Editorial Office <mathematics@mdpi.com>;Krystal Wang <krystal.wang@mdpi.com>

Dear Professor da Silva,

Congratulations on the acceptance of your manuscript, and thank you for submitting your work to Mathematics:

Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024
E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
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Kind regards,
Francisco Chiclana
Editor-in-Chief

[Mathematics] Manuscript ID: mathematics-2967439 - Funding Information Confirmation

leif.qi@mdpi.com <leif.qi@mdpi.com>
on behalf of
Mathematics Editorial Office <mathematics@mdpi.com>
Thu 4/25/2024 11:09 AM

To:Herry P Suryawan <herrypribs@usd.ac.id>;José L. da Silva <joses@staff.uma.pt>
Cc:Mathematics Editorial Office <mathematics@mdpi.com>;Krystal Wang <krystal.wang@mdpi.com>

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Fundação para a Ciência e Tecnologia: UIDB/MAT/04674/2020,
https://doi.org/10.54499/UIDB/04674/2020

Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024
E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
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Thu 4/25/2024 11:09 AM

To:Herry P Suryawan <herrypribs@usd.ac.id>
Cc:Krystal Wang <krystal.wang@mdpi.com>

Dear Professor Suryawan,

Congratulations on your paper being accepted for publication in Mathematics.

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Kind regards,

Dr. Shu-Kun Lin and the SciProfiles Team

[Mathematics] Manuscript ID: mathematics-2967439 - Final Proofreading Before Publication

yueyang.zhang@mdpi.com <yueyang.zhang@mdpi.com>
on behalf of
Mathematics Editorial Office <mathematics@mdpi.com>

Fri 4/26/2024 7:45 AM

To:José Luís Da Silva <joses@staff.uma.pt>
Cc:Herry P Suryawan <herrypribs@usd.ac.id>;Mathematics Editorial Office <mathematics@mdpi.com>; yueyang.zhang@mdpi.com <yueyang.zhang@mdpi.com>;Krystal Wang <krystal.wang@mdpi.com>

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Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024
E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
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We look forward to hearing from you soon.

Kind regards,
Ms. Yueyang Zhang
Production Editor
E-Mail: yueyang.zhang@mdpi.com

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[Mathematics] Manuscript ID: mathematics-2967439 - Manuscript Resubmitted

susy@mdpi.com <susy@mdpi.com>
on behalf of
Mathematics Editorial Office <mathematics@mdpi.com>

Fri 4/26/2024 10:02 PM

To:José Luís Da Silva <joses@staff.uma.pt>
Cc:Herry P Suryawan <herrypribs@usd.ac.id>

Dear Professor da Silva,

Thank you very much for resubmitting the modified version of the following manuscript:

Manuscript ID: mathematics-2967439
Type of manuscript: Article
Title: Green Measures for a Class of non-Markov Processes
Authors: Herry P. Suryawan, José L. da Silva *
Received: 31 Mar 2024
E-mails: herrypribs@usd.ac.id, joses@staff.uma.pt
Probability and Statistics
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Kind regards,
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Mon 4/29/2024 10:31 AM

To:Herry P Suryawan <herrypribs@usd.ac.id>;joses@staff.uma.pt <joses@staff.uma.pt>
Cc:mathematics@mdpi.com <mathematics@mdpi.com>;krystal.wang@mdpi.com <krystal.wang@mdpi.com>

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Keywords	fractional Brownian motion; generalized grey Brownian motion; green measure; subordination
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Authors Herry P. Suryawan , José L. da Silva * (https://sciprofiles.com/profile/1781880)

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Abstract In this paper, we investigate the Green measure for a class of non-Gaussian processes in $\mathbb{R}^{d}$. These measures are associated with the family of generalized grey Brownian motions $B_{\beta,\alpha}$, $0

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Comments and Suggestions for Authors

The paper "Green Measures for a Class of non-Markov Processes" by Suryawan and da Silva focuses on exploring Green measures associated with generalized grey Brownian motions (ggBm), extending the analysis to include both fractional Brownian motion and other non-Gaussian processes within a multi-dimensional setting. This research is significant in advancing the understanding of such processes in higher dimensions, particularly in dimensions where traditional methods do not necessarily apply or yield meaningful results.

The paper's introduction effectively sets the stage for the investigation of Green measures for non-Gaussian processes. However, explicitly delineating the novel contributions of this paper in comparison to existing literature early in the introduction would enhance its impact. Clarifying how this work extends beyond the boundaries of current knowledge, especially in the context of ggBm and its properties, would solidify the paper's significance.

The definitions and properties section provides a solid mathematical foundation for ggBm. It might be beneficial to include a more detailed discussion or examples illustrating the practical implications or physical phenomena that these processes model. This could help bridge the gap between theoretical underpinnings and real-world applications.

The core of the paper revolves around establishing the existence of the Green measure for ggBm and its explicit representation. Expanding on the significance of these findings, particularly how they relate to potential applications or how they advance the theoretical understanding of non-Markov processes, would add depth to the discussion. Insights into the implications of these results for stochastic modeling, physical sciences, or engineering would be valuable.

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Comments and Suggestions for Authors

This is an interesting paper that, however, requires certain modifications prior to its acceptance: the authors should reply in detail to the points listed below. The importance of the proposed measure must be clarified in much more detail. I do understand that it is a math-oriented paper, it is also formatted accordingly. Note however that the text is not easy to read for even-applied math people. The average reader would possibly wish somewhat more historical facts, a longer discussion of importance, comparison with other measures available in the literature for similar purposes, and---vitally---a longer discussion of possible applications of the proposed measure. The latter can particularly be added upon revision into a currently rather meager discussion section. The authors discuss BM, FBM, ggBM, but what about applicability of the proposed measure to other processes, possibly also of Gaussian nature of PDFs? For instance, what about a combination of FBM and SBM considered recently in ref. [https://doi.org/10.1039/D2CP01741E]? Such hybrid processes should be discussed in detail in the revised version: do they pose possible extensions of the applicability? If not, why? Or, e.g., what about geometric BM which is log-normal in its PDF? How well does this quantifier work for processes with a consecutive segments of different Gaussian-type processes, both Markovian and non-Markovian? See and mention here e.g. recent ref. [DOI: 10.1103/PhysRevResearch.6.013054].

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Are the conclusions supported by the results?	(x)	( )	( )	( )

Comments and Suggestions for Authors

The current revision is certainly improved and the material deserves to be published. I have only one general presentation-related concern, see eq. 16 for instance for the point below.

For a mathematically clear/clean presentation, all indices of quantities appearing on the left side of an equation should also appear on its right side. Notations should thus be chosen wisely in accord with this requirement. If, for instance, a combination of quantities or parameters containing indices is being redefined into a new parameter, the latter should contain all the indices it embracing as subscripts. This will make all later simplified equations look consistent with this rule.

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Article

Green Measures for a Class of non-Markov Processes

Herry P. Suryawan[1,‡], **José L. da Silva**[2,*]

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Abstract: In this paper, we investigate the Green measure for a class of non-Gaussian processes in $\mathbb{R}^d$. These measures are associated with the family of generalized grey Brownian motions $B_{\beta,\alpha}$, $0 < \beta \leq 1$, $0 < \alpha \leq 2$. This family includes both fractional Brownian motion, Brownian motion, and other non-Gaussian processes. We show that the perpetual integral exists with probability 1 for $d\alpha > 2$ and $1 < \alpha \leq 2$. The Green measure then generalizes those measures of all these classes.

Keywords: Fractional Brownian motion; generalized grey Brownian motion; Green measure; subordination

1. Introduction

In recent years, there has been a significant amount of research devoted to fractional dynamics related to fractional Brownian motion and related processes. These processes lack both the Markov and semimartingale properties from a mathematical standpoint. As a result, many traditional approaches in stochastic analysis do not apply, making their analysis more challenging. These processes are capable of modeling systems that exhibit long-range self-interaction and memory effects.
In 1992, Schneider introduced the grey Brownian motion [1], a class of non-Gaussian processes, to solve the time-fractional diffusion equation with Caputo-Djrbashian derivative of fractional order. During the 1990s, Mainardi and his co-authors conducted a systematic investigation into fractional differential equations; see [2] and references therein. They introduced the notion of generalized grey Brownian motion (ggBm for short), and the corresponding time-fractional differential equations governing its densities. This family of processes is denoted by $B_{\beta,\alpha}$ with parameters $0 < \beta \leq 1$ and $0 < \alpha \leq 2$. If $\beta \neq 1$ the process $B_{\beta,\alpha}$ is non-Gaussian with stationary increments and $\alpha/2$ -self-similar; see Section 2 for details. The process $B_{\beta,\alpha}$ admits different representations (cf. (11) and (12) below) in terms of other known processes, which are useful for simulation and to derive other properties. In a recent work, Grothaus et al. [3], elaborated an infinite dimensional analysis for (non-Gaussian) measures of Mittag-Leffler type. They used ggBm to solve the time-fractional heat equation extending the fractional Feynman-Kac from Schneider [1].

The goal of this paper (see Theorem 1 and Corollary 1 below) is to prove the existence of the Green measure for ~~a~~ the class of non-Gaussian processes ggBm in $\mathbb{R}^d$~~, called generalized grey Brownian motion (ggBm for short). We denote this family of processes by $B_{\beta,\alpha}$ with parameters $0 < \beta \leq 1$ and $0 < \alpha \leq 2$.~~. This result will extend the results of Kondratiev et al. [4]. More precisely, for a Borel function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, the potential of f (see [5,6] for details) is defined as

$$V_{\beta,\alpha}(f,x) = \int_0^\infty \mathbb{E}\big[f(x + B_{\beta,\alpha}(t))\big]\,\mathrm{d}t, \quad x \in \mathbb{R}^d. \tag{1}$$

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We would like to investigate the class of functions f for which the potential of f has the representation

$$V_{\beta,\alpha}(f,x) = \int_{\mathbb{R}^d} f(y)\mathcal{G}_{\beta,\alpha}(x,\mathrm{d}y), \tag{2}$$

where $\mathcal{G}(x,\cdot) := \mathcal{G}_{\beta,\alpha}(x,\cdot)$ is a Radon measure on $\mathbb{R}^d$ called Green measure corresponding to the ggBm $B_{\beta,\alpha}$, see Definition 2 below. If $B_{\beta,\alpha}$ admits a generator $L_{\beta,\alpha}$, then the potential $V(x,f)$ can be obtained from the equation

$$-LV = f.$$

The Green measure can be seen as the fundamental solution for the generator $L_{\beta,\alpha}$ of the process $B_{\beta,\alpha}$. First, we establish the existence of the perpetual integral (cf. Theorem 1)

$$\int_0^\infty f(x + B_{\beta,\alpha}(t))\,\mathrm{d}t$$

with probability 1. This leads to an explicit representation of the Green measure for ggBm, namely (cf. Corollary 1)

$$\mathcal{G}_{\beta,\alpha}(x,\mathrm{d}y) = \frac{D}{|x-y|^{d-2/\alpha}}\,\mathrm{d}y, \quad d\alpha > 2, \quad 1 < \alpha \leq 2,$$

where D is a constant that depends on β, α, and the dimension d; see ~~Theorem 1~~ (16) for the explicit expression. Note that as $d\alpha > 2$ and $1 < \alpha \leq 2$, the Green measure $\mathcal{G}_{\beta,\alpha}(x,\cdot)$ exists for $d \geq 2$, since $d > 2/\alpha \in [1,2)$. The Brownian case ($\alpha = 1$) is covered only for $d \geq 3$.

We emphasize that the existence of the Green measure for a given process X is not always guaranteed. In addition, finding a proper space of functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ that guarantee the existence of (1) is crucial. As an example, the d-dimensional ~~Brownian motion (Bm)~~ Bm starting at $x \in \mathbb{R}^d$ has a density given by $p_t(x,y) = (2\pi t)^{-d/2} \exp\big(-|x-y|^2/(2t)\big)$, $y \in \mathbb{R}^d$. It is not difficult to see that $\int_0^\infty p_t(x,y)\,\mathrm{d}t$ does not exist for $d = 1,2$. ~~This implies the non-existence of the~~ Hence, the Green measure of Bm for $d = 1,2$ does not exist. On the other hand, for $d \geq 3$, the Green measure of Bm on $\mathbb{R}^d$ exists and is given by $\mathcal{G}(x,\mathrm{d}y) = C(d)|x-y|^{2-d}\,\mathrm{d}y$, where $C(d)$ is a constant depending on the dimension d; see [4] and the references therein for more details. In a two-dimensional space, the Green measure of ggBm is determined by the parameter α that is related to the roughness of the path. The Green measure of ggBm for $d = 1$ requires further analysis (for Bm see [7], Ch. 4) which we will postpone for a future paper.

The paper is organized as follows. In Section 2 we recall the definition and main properties of ggBm that will be needed later. In Section 3 we show the existence of the perpetual integral with probability 1, which leads to the explicit formula for the Green measure for ggBm. In Section 4, we discuss the results obtained, connect them with other topics, and draw conclusions.

2. Generalized Grey Brownian Motion

We recall the class of non-Gaussian processes, called the generalized grey Brownian motion, which we study below. This class of processes was first introduced by Schneider [8,9], and was generalized by Mura et al. (see [10,11]) as a stochastic model for slow/fast anomalous diffusion described by the time fractional diffusion equation.

2.1. Definition and Properties

For $0 < \beta \leq 1$ the Mittag-Leffler (entire) function E_β is defined by the Taylor series

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}, \quad z \in \mathbb{C}, \tag{3}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad z \in \mathbb{C},\ \mathrm{Re}(z) \geq 0$$

is the Euler gamma function.

The M-Wright function is a special case of the class of Wright functions $W_{\lambda,\mu}$, $\lambda > -1$, $\mu \in \mathbb{C}$, via

$$M_\beta(z) := W_{-\beta,1-\beta}(-z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(-\beta n + 1 - \beta)}.$$

The special choice $\beta = 1/2$ yields the Gaussian density on $[0, \infty)$

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right). \tag{4}$$

The Mittag-Leffler function E_β is the Laplace transform of the M-Wright function, that is,

$$E_\beta(-s) = \int_0^\infty e^{-s\tau} M_\beta(\tau)\, d\tau. \tag{5}$$

The generalized moments of the density M_β of order $\delta > -1$ are finite and are given (see [10]) by

$$\int_0^\infty \tau^\delta M_\beta(\tau)\, d\tau = \frac{\Gamma(\delta + 1)}{\Gamma(\beta\delta + 1)}. \tag{6}$$

Definition 1. *Let $0 < \beta \leq 1$ and $0 < \alpha \leq 2$ be given. A d-dimensional continuous stochastic process $B_{\beta,\alpha} = \{B_{\beta,\alpha}(t),\ t \geq 0\}$ starting at $0 \in \mathbb{R}^d$ and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is a ggBm in $\mathbb{R}^d$ (see [11] for $d = 1$) if:*

1. *$\mathbb{P}(B_{\beta,\alpha}(0) = 0) = 1$, that is, $B_{\beta,\alpha}$ starts at zero $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.).*
2. *Any collection $\{B_{\beta,\alpha}(t_1), \ldots, B_{\beta,\alpha}(t_n)\}$ with $0 \leq t_1 < t_2 < \cdots < t_n < \infty$ has a characteristic function given, for any $\theta = (\theta_1, \ldots, \theta_n) \in (\mathbb{R}^d)^n$ with $\theta_k = (\theta_{k,1}, \ldots \theta_{k,d})$, $k = 1, \ldots, n$, by*

$$\mathbb{E}\left[\exp\left(\mathrm{i} \sum_{k=1}^n (\theta_k, B_{\beta,\alpha}(t_k))_{\mathbb{R}^d}\right)\right] = E_\beta\left[-\frac{1}{2} \sum_{j=1}^d (\theta_{\cdot,j}, \gamma_\alpha \theta_{\cdot,j})_{\mathbb{R}^n}\right], \tag{7}$$

where $\mathbb{E}$ denotes the expectation w.r.t. $\mathbb{P}$ and

$$\gamma_\alpha := \gamma_{\alpha,n} := \left(t_k^\alpha + t_j^\alpha - |t_k - t_j|^\alpha\right)_{k,j=1}^n.$$

3. *The joint probability density function of $(B_{\beta,\alpha}(t_1), \ldots, B_{\beta,\alpha}(t_n))$ is equal to*

$$\rho_\beta(\theta, \gamma_\alpha) = \frac{(2\pi)^{-\frac{nd}{2}}}{(\det\gamma_\alpha)^{d/2}} \int_0^\infty \tau^{-\frac{nd}{2}} \mathrm{e}^{-\frac{1}{2\tau} \sum_{j=1}^d (\theta_{\cdot,j}, \gamma_\alpha^{-1} \theta_{\cdot,j})_{\mathbb{R}^n}} M_\beta(\tau)\, d\tau. \tag{8}$$

The following are the most important key properties of ggBm:

(P1). For each $t \geq 0$, the moments of any order of $B_{\beta,\alpha}(t)$ are given by

$$\begin{cases} \mathbb{E}\left[|B_{\beta,\alpha}(t)|^{2n+1}\right] & = 0, \\ \mathbb{E}\left[|B_{\beta,\alpha}(t)|^{2n}\right] & = \frac{(2n)!}{2^n \Gamma(\beta n + 1)} t^{\alpha n}. \end{cases}$$

(P2). The covariance function has the form

$$\mathbb{E}\left[(B_{\beta,\alpha}(t), B_{\beta,\alpha}(s))\right] = \frac{d}{2\Gamma(\beta + 1)} \left(t^\alpha + s^\alpha - |t - s|^\alpha\right), \quad t, s \geq 0. \tag{9}$$

(P3). For each $t, s \geq 0$, the characteristic function of the increments is

$$\mathbb{E}\big[e^{i(k, B_{\beta,\alpha}(t) - B_{\beta,\alpha}(s))}\big] = E_\beta\left(-\frac{|k|^2}{2}|t-s|^\alpha\right), \quad k \in \mathbb{R}^d. \tag{10}$$

(P4). The process $B_{\beta,\alpha}$ is non-Gaussian, $\alpha/2$-self-similar with stationary increments.

(P5). The ggBm is not a semimartingale. Furthermore, $B_{\alpha,\beta}$ cannot be of finite variation in $[0, 1]$ and, by scaling and stationarity of the increment, on any interval in $\mathbb{R}^+$.

(P5). For $n = 1$, the density $\rho_\beta(x, t)$, $x \in \mathbb{R}^d$, $t > 0$, is the fundamental solution of the following fractional differential equation (see [12])

$$\mathbb{D}_t^{2\beta} \rho_\beta(x, t) = \Delta_x \rho_\beta(x, t),$$

where Δ_x is the d-dimensional Laplacian in x and $\mathbb{D}_t^{2\beta}$ is the Caputo-Dzherbashian fractional derivative; see [13] for the definition and properties.

2.2. Representations of Generalized Grey Brownian Motion

The ggBm admits different representations in terms of well-known processes. It follows from (7) that ggBm has an elliptical distribution, see Section 2 in [14] or Section 3 in [15]. On the other hand, ggBm is also given as a product (see [10] for $d = 1$) of two processes as follows

$$\{B_{\beta,\alpha}(t),\, t \geq 0\} \stackrel{\mathcal{L}}{=} \{\sqrt{Y_\beta} B^{\alpha/2}(t),\, t \geq 0\}. \tag{11}$$

Here, $\stackrel{\mathcal{L}}{=}$ means equality in law, the nonnegative random variable Y_β has density M_β and $B^{\alpha/2}$ is a d-dimensional fBm with Hurst parameter $\alpha/2$ and independent of Y_β.

We give another representation of ggBm $B_{\beta,\alpha}$ as a subordination of fBm (see Prop. 2.14 in [16] for $d = 1$) which is used below. For completeness, we give the short proof.

Proposition 1. *The ggBm has the following representation*

$$\{B_{\beta,\alpha}(t),\, t \geq 0\} \stackrel{\mathcal{L}}{=} \{B^{\alpha/2}(tY_\beta^{1/\alpha}),\, t \geq 0\}. \tag{12}$$

Proof. We must show that both representations (11) and (12) have the same finite-dimensional distribution. For every $\theta = (\theta_1, \ldots, \theta_n) \in (\mathbb{R}^d)^n$, we have

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n \left(\theta_k, B^{\alpha/2}(t_k Y_\beta^{1/\alpha})\right)\right)\right] &= \int_0^\infty \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n \left(\theta_k, B^{\alpha/2}(t_k y^{1/\alpha})\right)\right)\right] M_\beta(y)\,\mathrm{d}y \\
&= \int_0^\infty \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n \left(\theta_k, y^{1/2} B^{\alpha/2}(t_k)\right)\right)\right] M_\beta(y)\,\mathrm{d}y \\
&= \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{k=1}^n \left(\theta_k, Y_\beta^{1/2} B^{\alpha/2}(t_k)\right)\right)\right].
\end{aligned}$$

In the second equality, we used the $\alpha/2$-self-similarity of fBm. This completes the proof. □

3. The Green Measure for Generalized Grey Brownian Motion

In this section we show the existence of the Green measure for the ggBm, see (1) and (2). Let us begin by discussing the existence of the Green measure for a general stochastic process X.

Let $X = \{X(t),\ t \geq 0\}$ be a stochastic process in $\mathbb{R}^d$ starting from $x \in \mathbb{R}^d$. If $X(t)$, $t \geq 0$, has a probability distribution $\rho_{X(t)}(x, \cdot)$, then Eq. (1) becomes

$$V_X(x, f) = \int_0^\infty \int_{\mathbb{R}^d} f(y)\, \rho_{X(t)}(x, \mathrm{d}y)\, \mathrm{d}t. \tag{13}$$

Then, applying the Fubini theorem, the Green measure $\mathcal{G}_X(x, \cdot)$ of X is given by

$$\mathcal{G}_X(x, \mathrm{d}y) = \int_0^\infty \rho_{X(t)}(x, \mathrm{d}y)\, \mathrm{d}t,$$

assuming the existence of $\mathcal{G}_X(x, \cdot)$ as a Radon measure on $\mathbb{R}^d$. That is, for every bounded Borel set $B \in \mathcal{B}_b(\mathbb{R}^d)$ we have

$$\mathcal{G}_X(x, B) = \int_0^\infty \rho_{X(t)}(x, B)\, \mathrm{d}t < \infty.$$

If the probability distribution $\rho_{X(t)}(x, \cdot)$ is also absolutely continuous with respect to the Lebesgue measure, say $\rho_{X(t)}(x, \mathrm{d}y) = \rho_t(x, y)\, \mathrm{d}y$, then the function

$$g_X(x, y) := \int_0^\infty \rho_t(x, y)\, \mathrm{d}t, \quad \forall y \in \mathbb{R}^d, \tag{14}$$

is called the Green function of the stochastic process X. Moreover, the Green measure in this case is given by $\mathcal{G}_X(x, \mathrm{d}y) = g_X(x, y)\, \mathrm{d}y$.

This leads us to the following definition of the Green measure of a stochastic process X.

Definition 2. *Let $X = \{X(t),\ t \geq 0\}$ be a stochastic process on $\mathbb{R}^d$ starting from $x \in \mathbb{R}^d$ and $\rho_{X(t)}(x, \cdot)$ be the probability distribution of $X(t)$, $t \geq 0$. The Green measure of X is defined as a Radon measure on $\mathbb{R}^d$ by*

$$\mathcal{G}_X(x, B) := \int_0^\infty \rho_{X(t)}(x, B)\, \mathrm{d}t, \quad B \in \mathcal{B}_b(\mathbb{R}^d),$$

or

$$\int_{\mathbb{R}^d} f(y) \mathcal{G}_X(x, \mathrm{d}y) = \int_{\mathbb{R}^d} f(y) \int_0^\infty \rho_{X(t)}(x, dy)\, \mathrm{d}t, \quad f \in C_0(\mathbb{R}^d)$$

whenever these integrals exist.

In other words, $\mathcal{G}_X(x, B)$ is the expected length of time the process remains in B.

~~In order to~~ To state the main theorem which establishes the existence of the Green measure for ggBm, first, we introduce a proper Banach space of functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ such that

$$\int_0^\infty f\big(x + B_{\beta,\alpha}(t)\big)\, \mathrm{d}t$$

is finite $\mathbb{P}$-a.s. Without loss of generality, we may assume that $f \geq 0$ above. We define the space $CL(\mathbb{R}^d)$ of continuous real-valued on $\mathbb{R}^d$ by

$$CL(\mathbb{R}^d) := \big\{ f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ is continuous, bounded and } f \in L^1(\mathbb{R}^d) \big\}.$$

The space $CL(\mathbb{R}^d)$ becomes a Banach space with the norm

$$\|f\|_{CL} := \|f\|_\infty + \|f\|_1, \quad \forall f \in CL(\mathbb{R}^d),$$

where $\|\cdot\|_\infty$ denotes the sup-norm and $\|\cdot\|_1$ is the norm in $L^1(\mathbb{R}^d)$. The choice of $CL(\mathbb{R}^d)$ allows us to show that the family of random variables (also known as perpetual integral functionals)

$$\int_0^\infty f\big(x + B_{\beta,\alpha}(t)\big)\,\mathrm{d}t, \quad f \in CL(\mathbb{R}^d)$$

have finite expectations $\mathbb{P}$-a.s.

Theorem 1. *Let $f \in CL(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ be given and consider ggBm $B_{\beta,\alpha}$ with $d\alpha > 2$ and $1 < \alpha \le 2$. Then, the perpetual integral functional $\int_0^\infty f(x + B(t))\,\mathrm{d}t$ is finite $\mathbb{P}$-a.s. and its expectation equals*

$$\mathbb{E}\left[\int_0^\infty f(x + B_{\beta,\alpha}(t))\,\mathrm{d}t\right] = D\int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}}\,\mathrm{d}y, \tag{15}$$

where ~~$D = D(\beta,\alpha,d) = \frac{1}{\alpha}2^{-1/\alpha}\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2} - \frac{1}{\alpha}\right)\frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha})}$.~~

$$D = D(\beta,\alpha,d) = \frac{1}{\alpha}2^{-1/\alpha}\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2} - \frac{1}{\alpha}\right)\frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha})}. \tag{16}$$

Proof. Given $x \in \mathbb{R}^d$ and $f \in CL(\mathbb{R}^d)$ non-negative, let $\rho_\beta(\cdot, t^\alpha)$ denote the density of $B_{\beta,\alpha}(t)$, $t \ge 0$, given by (see (8) with $n = 1$)

$$\rho_\beta(y, t^\alpha) = \frac{1}{(2\pi t^\alpha)^{d/2}}\int_0^\infty \tau^{-d/2}\mathrm{e}^{-\frac{|y|^2}{2t^\alpha\tau}}M_\beta(\tau)\,\mathrm{d}\tau, \quad y \in \mathbb{R}^d.$$

First, we show the equality (15). It follows from the above considerations that

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty f(x + B_{\beta,\alpha}(t))\,\mathrm{d}t\right] &= \int_0^\infty \int_{\mathbb{R}^d} f(x+y)\rho_t^{\beta,\alpha}(y)\,\mathrm{d}y\,\mathrm{d}t.\\ &= \int_0^\infty \int_{\mathbb{R}^d} f(x+y)\frac{1}{(2\pi t^\alpha)^{d/2}}\int_0^\infty \tau^{-d/2}M_\beta(\tau)e^{-\frac{|y|^2}{2t^\alpha\tau}}\,\mathrm{d}\tau\,\mathrm{d}y\,\mathrm{d}t.\end{aligned}$$

Using Fubini's Theorem, we first compute the t-integral and use the assumption $d\alpha > 2$. We obtain

$$\int_0^\infty \frac{1}{(2\pi t^\alpha\tau)^{d/2}}e^{-\frac{|y|^2}{2t^\alpha\tau}}\,\mathrm{d}t = C(\alpha,d)\frac{\tau^{-\frac{1}{\alpha}}}{|y|^{d-2/\alpha}},$$

where

$$C(\alpha,d) := \frac{1}{\alpha}2^{-1/\alpha}\pi^{-\frac{d}{2}}\Gamma\left(\frac{d}{2} - \frac{1}{\alpha}\right).$$

Next we compute the τ-integral using (6) so that

$$\int_0^\infty \tau^{-1/\alpha}M_\beta(\tau)\,\mathrm{d}\tau = \frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha})}, \quad \alpha > 1.$$

Combining gives

$$\mathbb{E}\left[\int_0^\infty f(x + B_{\beta,\alpha}(t))\,\mathrm{d}t\right] = D\int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}}\,\mathrm{d}y,$$

where $D = D(\beta,\alpha,d) = C(\alpha,d)\frac{\Gamma(1-\frac{1}{\alpha})}{\Gamma(1-\frac{\beta}{\alpha})}$. Therefore, the equality (15) is shown.

Now we show that the right-hand side of (15) is finite for every non-negative $f \in CL(\mathbb{R}^d)$. To see this, we may use the local integrability of $|y|^{d-2/\alpha}$ in y and obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{f(x+y)}{|y|^{d-2/\alpha}}\,\mathrm{d}y &= \int_{\{|y|\le 1\}} \frac{f(x+y)}{|y|^{d-2/\alpha}}\,\mathrm{d}y + \int_{\{|y|>1\}} \frac{f(x+y)}{|y|^{d-2/\alpha}}\,\mathrm{d}y\\ &\le C_1\|f\|_\infty + C_2\|f\|_1 \le C\|f\|_{CL}.\end{aligned}$$

Therefore, the integral in (15) is, in fact, well defined. In other words, the integral $\int_0^\infty f(x+B_{\beta,\alpha}(t))\,\mathrm{d}t$ exists with probability 1. This completes the proof. □

As a consequence of the above theorem, we immediately obtain the Green measure of ggBm $B_{\beta,\alpha}$, that is, comparing (2) and (15).

Corollary 1. *The Green measure of ggBm $B_{\beta,\alpha}$ for $d\alpha > 2$ is given by*

$$\mathcal{G}_{\beta,\alpha}(x,\mathrm{d}y) = \frac{D}{|x-y|^{d-2/\alpha}}\,\mathrm{d}y,$$

where D is given by (16).

Remark 1. 1. *It is possible to show that given $f \neq 0$, the perpetual integral $\int_0^\infty f(x+B_{\beta,\alpha}(t))\,\mathrm{d}t$ is a non-constant random variable. As a consequence, for $f \ge 0$ the variance of $\int_0^\infty f(x+B_{\beta,\alpha}(t))\,\mathrm{d}t$ is strictly positive. The proof uses the notion of conditional full support of ggBm. We will not provide a detailed explanation of this result that closely follows the ideas of Theorem 2.2 in [4] to which we address the interested readers.*

2. *Note also that the functional in* (1)

$$V_{\beta,\alpha}(\cdot,x) : CL(\mathbb{R}^d) \longrightarrow \mathbb{R}$$

is continuous. In fact, from the proof of Theorem 1 for any $f \in CL(\mathbb{R}^d)$ yields

$$|V_{\beta,\alpha}(f,x)| \le K\|f\|_{CL},$$

where K is a constant depending on the parameters β, α, and d.

4. Discussion and Conclusions

We have derived the Green measure for the class of stochastic processes called the generalized grey Brownian motion in Euclidean space $\mathbb{R}^d$ for $d \ge 2$. This class includes, in particular, fractional Brownian motion and other non-Gaussian processes. To address the case where $d = 1$, a renormalization process is needed. However, this will be postponed to future work. For $\beta = \alpha = 1$ ggBm $B_{1,1}$ is nothing but a Brownian motion. In this case, the Green measure exists for $d \ge 3$. Green measures and Green functions are well known to be intrinsically connected and applied to (stochastic partial) differential equations. In this context, the Green measures discussed in this paper play the same role for space-time-fractional derivatives. The presented method used can be applied to other processes with sufficient information on the density and existence of the integrals. If we consider a Markov process X that admits a Green measure and T a random time change given by an inverse subordinator, then the Green measure of the subordinated process $X(T(t))$, $t \ge 0$, exists only after renormalization; see [17]. Mixing different type of processes, e.g. fBm and scaled Bm as described in [18] or Markovian and non-Markovian as in [19], may lead us to a renormalization procedure in order to guarantee the existence of the Green measure.

The relationship between the Green measure and the local time of the ggBm can be described as follows. For any $T > 0$ and a continuous function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, the integral functional

$$\int_0^T f(B_{\beta,\alpha}(t))\,\mathrm{d}t \tag{17}$$

is well-defined. For $d = 1$ the integral (17) with $f \in L^1(\mathbb{R})$ is represented as

$$\int_0^T f(B_{\beta,\alpha}(t))\,\mathrm{d}t = \int_{\mathbb{R}} f(x) L_{\beta,\alpha}(T,x)\,\mathrm{d}x,$$

where $L_{\beta,\alpha}(T,x)$ is the local time of ggBm up to time T at the point x, see [14,15]. The Green measure corresponds to the asymptotic behaviour in T of the expectation of local time $L_{\beta,\alpha}(T,x)$. The existence of this asymptotic depends on the dimension d and the transient or recurrent properties of the process.

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Response to Reviewer 2 Comments

1. Summary

We thank very much the referee for the comments, suggestions and questions addressed to the first version of the manuscript. Please find the detailed responses below and the corresponding revisions/corrections highlighted/in track changes in the re-submitted files.

2. Questions for General Evaluation	**Reviewer's Evaluation**	**Response and Revisions**
Does the introduction provide sufficient background and include all relevant references?	Can be improved	*We have improved largely the introduction providing more background, applications, and literature review.*
Are all the cited references relevant to the research?	Can be improved	We improved the list of relevant references cited in the body of manuscript.
Is the research design appropriate?	Can be improved	We improved the relevance of the results and discussed potential applications and connections
Are the methods adequately described?	Can be improved	We hope this revision improved the clarity and importance of the results.
Are the results clearly presented?	Can be improved	We improved and highlighted the new results of the manuscript.
Are the conclusions supported by the results?	Can be improved	The conclusions and discussions are improved.

3. Point-by-point response to Comments and Suggestions for Authors

Comments 1: This is an interesting paper that, however, requires certain modifications prior to its acceptance: the authors should reply in detail to the points listed below. The importance of the roposed measure must be clarified in much more detail. I do understand that it is a math-oriented paper, it is also formatted accordingly. Note however that the text is not easy to read for even-applied math people. The average reader would possibly wish somewhat more historical facts, a longer discussion of importance, comparison with other measures available in the literature for similar purposes, and---vitally---a longer discussion of possible applications of the proposed measure. The latter can particularly be added upon revision into a currently rather meager discussion section. The authors discuss BM, FBM, ggBM, but what about applicability of the proposed measure to other processes, possibly also of Gaussian nature of PDFs? For instance, what about a combination of FBM and SBM considered recently in ref. [https://doi.org/10.1039/D2CP01741E]? Such hybrid processes should be discussed in detail in the revised version: do they pose possible extensions of the applicability? If not, why? Or, e.g., what about geometric BM which is log-normal in its PDF? How well does this quantifier work for processes with a consecutive segments of different Gaussian-type processes, both Markovian and non-Markovian? See and mention here e.g. recent ref. [DOI: 10.1103/PhysRevResearch.6.013054].

Response 1: We agree with these comments and suggestions. Therefore, we have improved the paper namely in the Introduction (page 1 and 2) and Discussion and Conclusions, page 7. We have updated the list of references in pages 8-9. All changes are highlighted in the file GM-ggBm-v3-diff.pdf.

Response to Reviewer 1 Comments

1. Summary

We thank very much the referee for the comments, suggestions and questions addressed to the first version of the manuscript. Please find the detailed responses below and the corresponding revisions/corrections highlighted/in track changes in the re-submitted files.

2. Questions for General Evaluation	**Reviewer's Evaluation**	**Response and Revisions**
Does the introduction provide sufficient background and include all relevant references?	Can be improved	*We have improved largely the introduction providing more background, applications, and literature review.*
Are all the cited references relevant to the research?	Yes	We improved the list of relevant references cited in the body of manuscript.
Is the research design appropriate?	Yes	We improved the relevance of the results and discussed potential applications and connections
Are the methods adequately described?	Can be improved	We hope this revision improved the clarity and importance of the results.
Are the results clearly presented?	Must be improved	We improved and highlighted the new results of the manuscript.
Are the conclusions supported by the results?	Can be improved	The conclusions and discussions are improved.

3. Point-by-point response to Comments and Suggestions for Authors

Comments 1: The paper "Green Measures for a Class of non-Markov Processes" by Suryawan and da Silva focuses on exploring Green measures associated with generalized grey Brownian motions (ggBm), extending the analysis to include both fractional Brownian motion and other non-Gaussian processes within a multi-dimensional setting. This research is significant in advancing the understanding of such processes in higher dimensions, particularly in dimensions where traditional methods do not necessarily apply or yield meaningful results.

Response 1: We agree with this comment.

Comments 2: The paper's introduction effectively sets the stage for the investigation of Green easures for non-Gaussian processes. However, explicitly delineating the novel contributions of this paper in comparison to existing literature early in the introduction would enhance its impact. Clarifying how this work extends beyond the boundaries of current knowledge, especially in the context of ggBm and its properties, would solidify the paper's significance.

Response 2: We have, improved largely the introduction highlighting the applications, literature review and connections to other processes, see page 1, 7, 8-9.

Comments 3: The definitions and properties section provides a solid mathematical foundation for gBm. It might be beneficial to include a more detailed discussion or examples illustrating the practical implications or physical phenomena that these processes model. This could help bridge the gap between theoretical underpinnings and real-world applications.

Response 3: We provided relevant applications for these processes where Green measures can be useful, see page 1 and 7.

Comments 4: The core of the paper revolves around establishing the existence of the Green measure for ggBm and its explicit representation. Expanding on the significance of these findings, particularly how they relate to potential applications or how they advance the theoretical understanding of non-Markov processes, would add depth to the discussion. Insights into the implications of these results for stochastic modeling, physical sciences, or engineering would be valuable.

Response 4: In Section 4. Discussion and Conclusions we provide relevant discussions on the subject with connections and applications to other processes known in the literature, see page 7.

For review article

Response to Reviewer X Comments

1. Summary

Thank you very much for taking the time to review this manuscript. Please find the detailed responses below and the corresponding revisions/corrections highlighted/in track changes in the re-submitted files. *[This is only a recommended summary. Please feel free to adjust it. We do suggest maintaining a neutral tone and thanking the reviewers for their contribution although the comments may be negative or off-target. If you disagree with the reviewer's comments please include any concerns you may have in the letter to the Academic Editor.]*

2. Questions for General Evaluation	**Reviewer's Evaluation**	**Response and Revisions**

Is the work a significant contribution to the field?	★★★★☆	*[Please give your response if necessary. Or you can also give your corresponding response in the point-by-point response letter. The same as below]*
Is the work well organized and comprehensively described?	★★★☆☆	
Is the work scientifically sound and not misleading?	★★★★☆	
Are there appropriate and adequate references to related and previous work?	★★☆☆☆	
Is the English used correct and readable?	★★★☆☆	

3. Point-by-point response to Comments and Suggestions for Authors

Comments 1: *[Paste the full reviewer comment here.]*

Response 1: *[Type your response here and mark your revisions in red]* Thank you for pointing this out. I/We agree with this comment. Therefore, I/we have….*[Explain what change you have made. Mention exactly where in the revised manuscript this change can be found – page number, paragraph, and line.]*
"[updated text in the manuscript if necessary]"

Comments 2: *[Paste the full reviewer comment here.]*

Response 2: Agree. I/We have, accordingly, done/revised/changed/modified…..to emphasize this point. *Discuss the changes made, providing the necessary explanation/clarification. Mention exactly where in the revised manuscript this change can be found – page number, paragraph, and line.]*
"[updated text in the manuscript if necessary]"

4. Response to Comments on the Quality of English Language

Point 1:

Response 1: (in red)

5. Additional clarifications

[Here, mention any other clarifications you would like to provide to the journal editor/reviewer.]