

It's time to build skyscrapers: A digital task sequence on graphs as representations of covarying quantities

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Abstract

This study proposes a digital task sequence, drawing on instruction based on the foundational principles of duality, necessity, and repeated reasoning (DNR-based instruction), designed to support students in developing an understanding of graphs as representations of covarying quantities. It reports on a lesson case involving 22 first-year pre-service mathematics teachers who engaged with the sequence. The findings highlight the potential of the digital task sequence to develop emergent graphical shape thinking (EGST) as a productive way of thinking about graphs. Through tasks that progressively bridge situational quantitative and covariational reasoning and reasoning with graphical representations, students demonstrated their ability to interpret, construct, and refine meanings related to graphs as representations of covarying quantities.

Keywords

Graphs; emergent graphical shape thinking; intellectual need; DNR

INTRODUCTION

Graphs representing covarying quantities are a common tool used to convey information to informed citizens. From illustrating the rise in global sea levels over the years to depicting trends in teen health-related behaviours, graphs effectively convey important messages. Thus, interpreting and constructing graphs are essential skills for students to develop as both consumers and producers of information (Binali et al., 2024; OECD, 2023). However, researchers have reported students' difficulties with graphs (Glazer, 2011; Hadjidemetriou & Williams, 2002; Ivanjek et al., 2017; Van Dooren et al., 2008). In this study, we propose a digital task sequence designed to support students in developing an understanding of graphs as representations of covarying quantities.

Integrating theories for designing a digital task sequence

In designing the digital task sequence, we integrated two perspectives: the DNR (duality, necessity, and repeated reasoning) framework for mathematics curriculum and instruction (Harel, 2008a, 2008b) and emergent graphical shape thinking (EGST, Moore & Thompson, 2015; Paoletti et al., 2023). Drawing on Harel's DNR framework, we articulated a pedagogical stance for teaching and learning mathematics. Drawing on EGST, we promote a productive way of thinking that supports students in interpreting and constructing graphs.

The DNR framework outlines three foundational instructional principles, i.e. duality, necessity, and repeated reasoning (Harel, 2008a, 2008b). The duality principle is based on the view that mathematics consists of two intertwined sets of knowledge: ways of understanding and ways of thinking (Harel, 2008b). The duality principle suggests that students develop ways of thinking through the production of ways of understanding, and the ways of understanding they generate are influenced by the ways of thinking they hold (Harel, 2008a). The second principle is the necessity principle, which states that for students to learn the mathematics intended for instruction, they must have an intellectual need for it. Intellectual need refers to a problematic situation that arises when an individual encounters a situation incompatible with or unsolvable by their current knowledge (Harel, 2013). Lastly, the repeated reasoning principle states that students must engage in reasoning repeatedly to internalize desirable ways of understanding and thinking (Harel, 2008a).

Moore and Thompson (2015) distinguished between two forms of graphical shape thinking: static and emergent. The former refers to viewing a graph as an object in itself, while the latter involves envisioning a graph as being dynamically generated by the trace of a moving point that represents covarying quantities. Researchers emphasize that emergent graphical shape thinking (EGST) is crucial for understanding concepts

in advanced mathematics and other STEM disciplines (Paoletti et al., 2024; Thompson & Harel, 2021). Paoletti et al. (2023) developed a framework outlining the key components necessary for students to engage in EGST. According to this framework, students must first engage in situational quantitative reasoning and covariational reasoning as well as reasoning with graphical representations as prerequisites for developing EGST (Paoletti et al., 2024).

Digital task sequence

We applied the instructional design principles of duality, necessity, and repeated reasoning proposed by Harel (2008a, 2008b) in designing the digital task sequence. Guided by the duality principle, the task sequence emphasized both graphs and EGST as central to students’ activity. Drawing on the necessity principle, students were prompted to devise their strategies for effectively communicating about two covarying quantities. These student-generated strategies served as an entry point for introducing graphs as representations of covarying quantities. Finally, in alignment with the repeated reasoning principle, the task sequence provided multiple opportunities for students to engage in situational quantitative and covariational reasoning, reasoning with graphical representations, and EGST to interpret and construct graphs.

A digital task sequence titled “It’s Time to Build Skyscrapers” was developed using Desmos, which is publicly accessible at <https://s.id/build-skyscrapers>. The sequence consists of four tasks, which are detailed in Table 1.

Table 1: Description of the digital task sequence.

Task	Objective	Description
1	Students generate their own strategies to communicate covarying quantities through text and images.	Students first locate the positions of two buildings and provide reasons for their choices. Next, they drag to adjust the position of a car on the street. They then match the distances between the car and the two buildings with preconstructed segments (Figure 1(a)). Finally, they communicate the relationship between the distances using text and images.
2	Students analyse and evaluate the mathematical thinking and strategies of others to envision a graph both as the trace it creates and as a representation of covarying quantities.	Students analyse a strategy that uses a graph to represent the distances between the car and the two buildings (Figure 2(a)). By observing an animation, they interpreted the graph as dynamically generated by the trace of a moving point constrained by those two covarying quantities (Figure 2(b)). Finally, they define a graph in their own words.
3	Students explore different situations that can produce the same final graph through varying traces.	Given a segment of a graph, students identify different situations that could produce it (Figure 1(b)). They are then presented with another segment of the graph and repeat the process, identifying potential situations that could result in the new segment.
4	Students construct a graph to represent changing quantities within a given context.	Students are first presented with a situation involving a car moving along a straight path between two buildings. They construct a graph representing the distances between the car and the two buildings. Next, they are given a situation where the car moves along a non-straight path, and they construct a corresponding graph for the distances (Figure 1(c)).

Table 1 outlines how the digital task sequence aligns with the EGST framework (Paoletti et al., 2023). It offers opportunities for students to engage in both situational quantitative and covariational reasoning as well as reasoning with graphical representations and further supports engagement with EGST as students interpret and construct graphs.

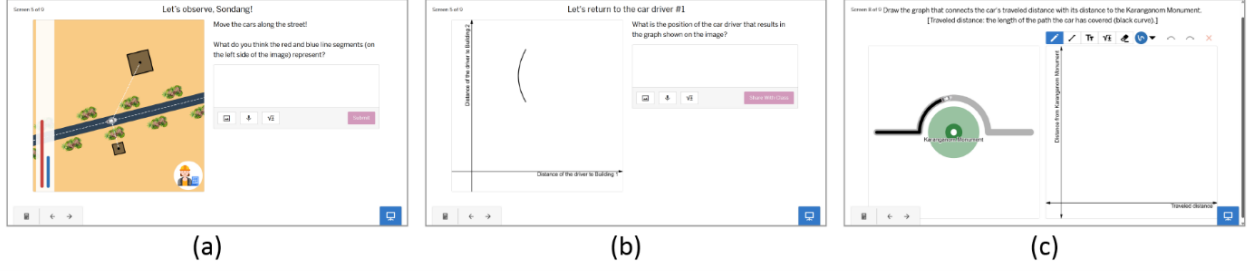


Figure 1: Screens from the digital task sequence.

Task 2 plays an important role in the digital task sequence, as it is the first point where students are expected to construct meaning for graphs as dynamically generated representations. In this task (see Figure 2(a)), students are intended to coordinate two varying quantities, specifically the distances between the car and each of the two buildings, and reason about how these quantities change in relation to each other. They then consider the corresponding variations in the lengths of two orthogonal segments on the coordinate axes, each representing one of the distances. By conceiving a point in the coordinate plane as a multiplicative object that simultaneously represents the magnitudes of both segments, students begin to build connections between the real-world situation and its graphical representation. These situational quantitative and covariational reasoning, along with reasoning with graphical representations, are intended to support students' meanings for EGST, namely their understanding of graphs as traces of moving points, dynamically generated through the interplay of covarying quantities (see Figure 2(b)). This understanding is foundational for engaging with subsequent tasks, where students are expected to interpret a given graph in Task 3 and construct graphs in Task 4.

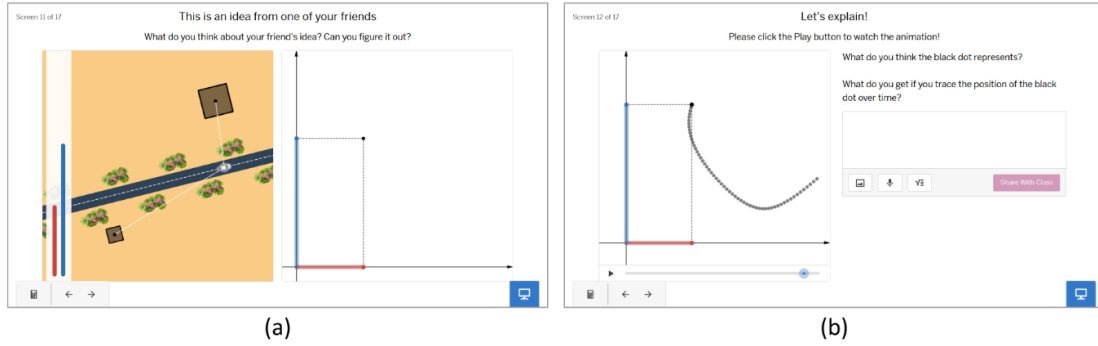


Figure 2: Screens from Task 2.

In summary, the digital task sequence was designed based on instructional principles: duality, to help students develop an understanding of graphs through EGST as a way of thinking; necessity, to present situations that naturally require the use of graphs; and repeated reasoning, to offer students multiple opportunities to revisit and deepen their understanding of graphs throughout the tasks. These principles guided the design of tasks that progressively engage students in reasoning about covarying quantities and in constructing a conceptual understanding of graphs as representations of these relationships.

METHODS

This study aims to propose a digital task sequence designed to support students in developing an understanding of graphs as representations of covarying quantities. The structure and design of the task sequence have been detailed in the preceding section. Accordingly, the central research question guiding this study is: How can the digital task sequence support students in developing EGST as a way of thinking for constructing an understanding of graphs?

This study is part of a larger design research project aimed at transforming mathematics curriculum and instruction related to change and relationships. Specifically, it presents a lesson case showcasing the implementation of a digital task sequence on graphs. The participants were 22 first-year pre-service mathematics teachers enrolled in an algebra and trigonometry course at a private university in Yogyakarta, Indonesia.

The lessons were conducted in two separate sessions, each lasting 100 minutes. The first session took place in late August 2023, followed by the second in early September 2023. Students worked in groups of two or three. In the first session, they completed the first task, generating their strategies to communicate covarying quantities using text and images. In the second session, students worked on the second, third, and fourth tasks.

The data for this study consists of students' work on the digital platform Desmos, which includes both textual and image-based responses that capture their answers and reasoning for the tasks. The analysis was guided by the EGST framework (Paoletti et al., 2023), which served as a lens to examine how the digital task sequence supported students in developing EGST as a way of thinking. The framework consists of situational quantitative and covariational reasoning (M.S.), reasoning with graphical representations of covarying quantities (M.R.), and EGST (M.E.). Using this framework, the first author coded students' works on the digital platform. The coded data were then discussed with the research team to ensure agreement and consistency in the interpretation.

RESULTS AND DISCUSSION

In this section, we present students' work in response to the digital task sequence. We then interpret their responses to examine how the opportunities provided by the sequence supported the development of EGST and contributed to their understanding of graphs as representations of covarying quantities.

In the first task, students developed various strategies to communicate covarying quantities. All strategies involved sketching a literal representation of the situation, depicting a car on the street and two buildings. However, six out of ten groups represented a dynamic process in their sketches by illustrating two or more positions of the car along the street. Figure 3 shows Group 1's work, which illustrates this approach by intentionally selecting two key positions of the car to convey their message.

The students' response in Figure 3 demonstrates that they developed meanings related to M.S. and M.R. They were able to identify the covarying quantities in the given situation and represent their magnitudes through varying segment lengths. Additionally, they coordinated how these quantities change in relation to each other. They divided the situation into three cases based on two key points: the first point, where the car is directly in front of building 2, resulting in the smallest distance between the car and building 2, and the second point, where the car is directly in front of building 1, resulting in the smallest distance between the car and building 1. From their answer (Figure 3), we infer that the students developed an operative image of covariation, at the stage of directional covariation (Carlson et al., 2002).

In Task 2, the students were introduced to a strategy for conveying the relationship between the two distances, i.e. the distance between the car and building 1, and the distance between the car and building 2, using a graph on a rectangular coordinate plane. When asked to analyse the graph, two groups, i.e. Group 1 and 9, demonstrated meanings related to M.E. Below is a translated excerpt of Group 1's interpretation of the graph.

The black point represents the driver's distances from both buildings. The vertical position of the black point represents the driver's distance to the second building. The horizontal position of the black point represents the driver's distance to the first building. If the driver moves from the west until directly in front of the second building, the black point will move to the bottom left. If the driver moves from in front of the second building until directly in front of the first building, the black dot will move to the top left.

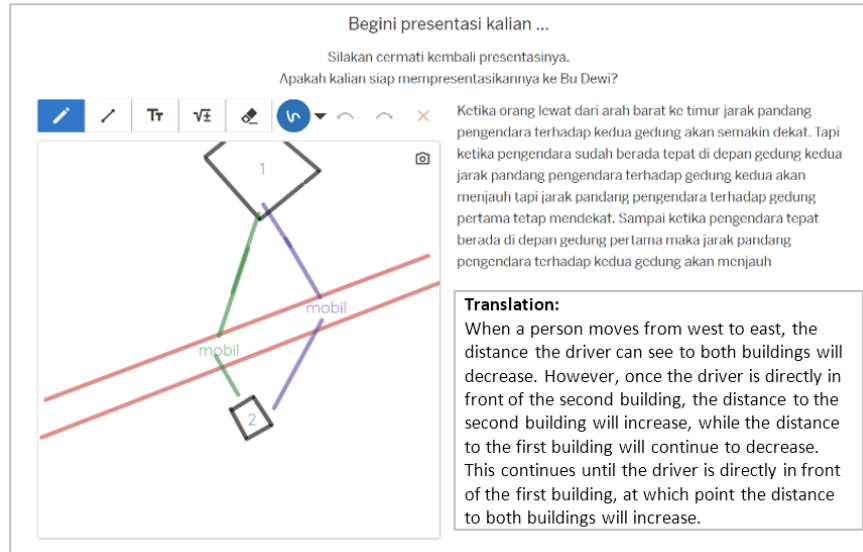


Figure 3: Group 1's response to Task 1.

From this group's interpretation of the graph, we infer that they perceived the motion of a point in the coordinate plane as representing changes in the distances. They also conceptualized the graph as the trace of the point, illustrating how the distances covary.

However, eight groups did not succeed in constructing meanings related to M.E. Group 2, for instance, struggled to interpret the point's movement, as shown in the following translated excerpt.

If the black point moves upward, the driver gets closer to the larger building. If the black point moves downward, the driver gets closer to the smaller building.

Based on the excerpt, it appears that Group 2 interpreted the point on the coordinate plane as representing the driver's literal position. As a result, they stated that the higher the point, the closer the driver is to the larger building, and the lower the point, the closer the driver is to the smaller building.

In Task 3, two groups, i.e. Group 3 and Group 10, successfully identified different situations that could produce the given graph. Figure 4 shows that Group 3 was able to identify two distinct situations represented by the given graph in the first question of Task 3.

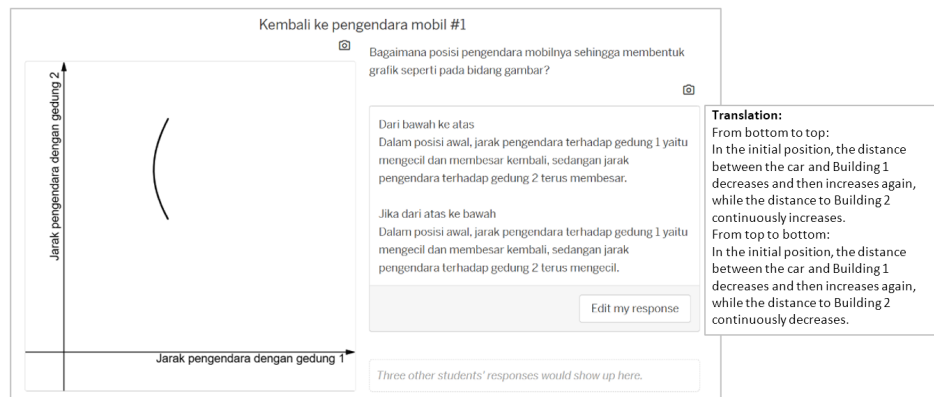


Figure 4: Group 3's response to the first question of the Task 3.

Group 3's response in Figure 4 demonstrates their attempt to describe the relationships between the car's position and its distances to the two buildings. They distinguished two directional movements along the graph, i.e. from bottom to top and from top to bottom and explained how the distances change in each case. From this response, we infer that the group demonstrated meanings related to M.E., as they identified two distinct situations that could produce the given graph by interpreting its traces in different directions.

Five groups interpreted the graph in the first question of Task 3 in only one direction. Group 1, for instance, analysed the graph from the bottom-to-top direction. Their response is presented in the following translated excerpt.

If the point moves to the upper left, it indicates that the car is moving closer to the first building and farther from the second building until a certain point. The point will then move to the upper right, indicating that the car is moving away from both buildings.

In Task 4, most groups successfully answered the first question, constructing a graph to represent covarying quantities from the given situation. A similar pattern was observed in their responses to the second question, which is illustrated in Figure 1(c). Figure 5 displays all groups' answers to the second question.

Figure 5 illustrates that most groups (8 out of 10) constructed appropriate graphs in response to the second question of Task 4, accurately representing the given situation. Their use of linear relationships for segments of their graphs appears appropriate to the context. However, the students' sketches do not provide enough evidence to infer their reasoning behind this choice.

We identified the progression of students' EGST as they engaged with the task sequence. The problem of conveying information about covarying quantities served as a bridge to introduce graphs. The ability of graphs to effectively communicate covarying quantities provided students with a clear rationale to understand and utilize graphs. Using Harel's term (Harel, 2013, 2024), this allowed students to perceive an epistemological justification for the necessity of graphs. These features reflect key principles of project-based instruction, particularly the use of a strong guiding problem and the active engagement of students (Rusek & Becker, 2011) in exploring and constructing mathematical ideas within a purposeful context.

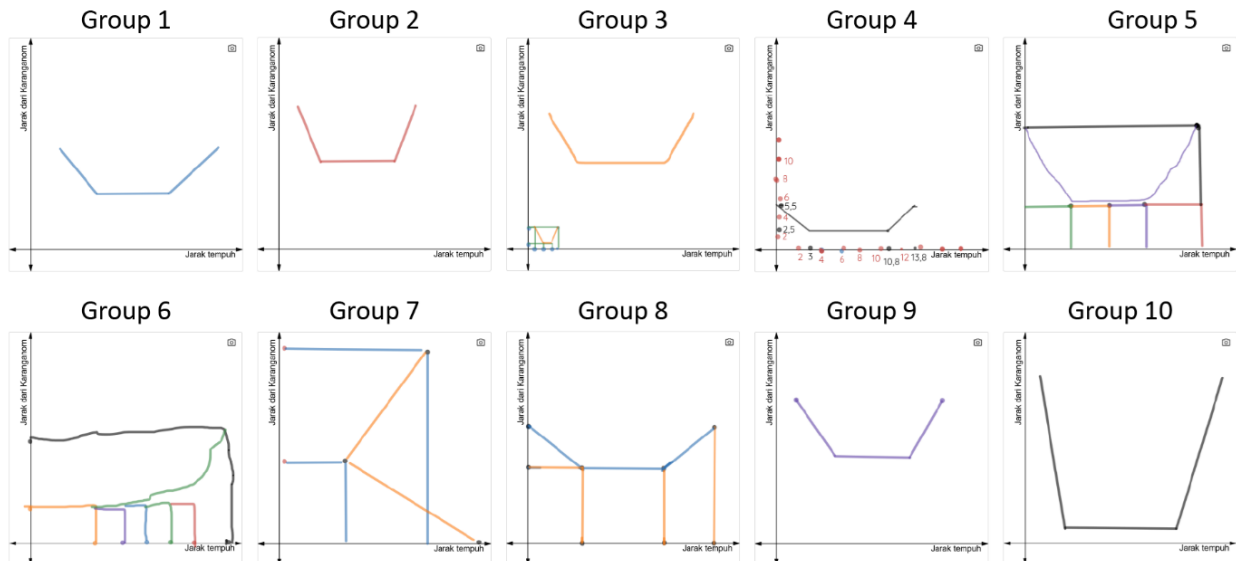


Figure 5: Sketches created by all groups in response to the second question of Task 4.

We found that most students were not successful in completing Task 2, which is particularly noteworthy given that this task was designed to serve as a conceptual foundation of graphs for the subsequent activities. Drawing on cognitive load theory (Sweller, 1988; Sweller et al., 2019), we suspect that this difficulty may

be attributed to a split-attention effect, as students were required to integrate related information presented across two separate screens (see Figure 2(a) and 2(b)). To address this, we plan to revise the task into a single-screen format that presents both visualizations concurrently. This strategy is supported by previous research showing that integrated designs can reduce cognitive load (Kueker & Moore, 2024). Additionally, we will incorporate the segmentation principle to manage cognitive load by gradually introducing information (Lusk et al., 2009). As a result, in the revised task, students will first see the changing lengths of orthogonal segments on each axis, which represent the distances to the two buildings, as they manipulate the car on the street. Next, they will be shown a point on the coordinate plane that corresponds to these distances. Finally, they will observe how the trace of that point forms a graph, allowing for a more coherent and scaffolded understanding of the covarying relationship.

We also found that providing students with repeated opportunities to engage in situational quantitative and covariational reasoning, reasoning with graphical representations of covarying quantities, and EGST, helped them refine their understanding of graphs. This result aligns with findings from previous studies (Moore, 2014; Paoletti et al., 2024). For example, Group 2, which initially struggled with interpreting graphs in Task 2, was able to construct meaningful interpretations of graphs by Task 4. These findings highlight the importance of repeated practice for developing fluency, aligning with conclusions from other studies (Harel, 2021; Soto et al., 2022). However, we acknowledge that some students were still unable to construct graphs successfully by the end of the task sequence. This raises important questions about how to best support students in developing EGST through repeated reasoning: How many tasks are needed for students to achieve understanding? What types of tasks, and how should they be sequenced? These questions offer valuable insight for further studies. For example, further studies could incorporate tasks that encourage the use of multiple representations for interpreting and constructing graphs, as suggested by previous studies (Johnson, 2022; Moore et al., 2013).

The findings of this study underscore the significance of EGST as a productive way of thinking for students to develop a meaningful understanding of graphs. This study offers a lens to organize and frame curricular approaches to teaching graphs and their functions. For example, we align with Moore and Thompson’s (2015) hypothesis on function transformations, suggesting that a solid understanding of graphs as representations of covarying quantities is essential for students to develop a meaningful and productive understanding of function transformations.

While this study provides valuable insights into students’ learning through the digital task sequence, there are some limitations to consider. First, the methods relied on interpreting students’ learning based on their works in the platform, which offers limited evidence and does not capture a complete picture of their cognitive processes. Second, the technology used for the digital task sequence is primarily optimized for tablets and computers, and its functionality is limited on mobile devices. This restricts its potential for widespread use in schools, where access to appropriate devices may vary.

CONCLUSION

The findings of this study highlight the potential of the digital task sequence to support students in developing EGST as a productive way of thinking about graphs. Through tasks that progressively bridge situational quantitative and covariational reasoning and reasoning with graphical representations, students demonstrated their ability to interpret, construct, and refine meanings related to graphs as representations of covarying quantities. While many students successfully developed a robust understanding of graphs, the study also identified challenges faced by some students, underscoring the importance of providing repeated opportunities and carefully sequenced tasks to support the development of fluency.

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