Students' meaning-making of horizontal function translation with a dynamic visualization

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Previous studies have shown that students often struggle with the horizontal translation of a function, and these difficulties have motivated subsequent research to propose instructional approaches either to address the challenges or to support students in developing more productive ways of thinking about this transformation. This study investigates how students can use emergent graphical shape thinking (EGST) to construct meanings for horizontal function translation when supported by a dynamic visualization. A case study approach was employed to trace the learning progression of two students as they worked through a digital task incorporating this visualization. Their progression unfolded in three phases. First, they identified a relationship between the original function and its horizontal translation, though the direction of this relationship was the reverse of what was intended. Second, in an effort to refine the relationship between the functions, they established a correspondence relationship between the inputs of the original and translated functions. Finally, they used this correspondence relationship to generalize the functional relationship that characterizes a horizontal translation. The ways in which EGST and the dynamic visualization supported this meaning-making are subsequently discussed.

Keywords: dynamic visualizations, emergent graphical shape thinking, function transformations, graph reasoning, horizontal function translation

Introduction

Previous studies have shown that students often struggle with the horizontal translation of a function (Baker et al., 2001; Zazkis et al., 2003), and these difficulties have motivated subsequent research to propose instructional approaches either to address the challenges or to support students in developing more productive ways of thinking about this transformation (Yim & Lee, 2022; Zazkis et al., 2003). Researchers have attributed students' challenges to the inherent complexity of horizontal translation (Baker et al., 2001), the counterintuitive graphical effect of its algebraic rule (Yim & Lee, 2022), students' unproductive meaning of functions (Baker et al., 2001; Lage & Gaisman, 2006), and various instructional obstacles (Yao, 2024; Zazkis et al., 2003). This body of work has led to a range of suggestions for teaching horizontal translation, from rethinking how students conceptualize functions to drawing on multiple representations. In terms of reconceptualizing functions, Thompson and colleagues have argued for emphasizing a process view of function and supporting students' covariational reasoning (Oehrtman et al., 2008; Thompson & Carlson, 2017). In work focusing on multiple representations, many researchers have highlighted the affordances of technology for helping students make sense of function transformations (Borba & Confrey, 1996; Božić et al., 2021; Ross et al., 2011).

In the same vein, this study draws on the affordances of technology to present multiple representations of a function, namely its graph and its algebraic representation, while also promoting a dynamic conceptualization of a function as a covariational relationship between two quantities. In doing so, the study foregrounds the kinds of productive meanings students can develop for function graphs and their transformations, and how technology, particularly dynamic visualization, can support that meaning-making. To frame this work, we use emergent graphical shape thinking as the productive way of thinking we aim to foster in students as they conceive function graphs and their transformations. We also draw on frameworks for interactive dynamic visualizations to guide the design of tasks that support students' construction of meaning for horizontal function translation. In this study, we use meaning as defined by Thompson (2015), referring to the collection of actions, objects, or schemes an individual draws on when assimilating a situation.

Theoretical frameworks

Emergent graphical shape thinking

Emergent graphical shape thinking (EGST) involves conceptualizing a graph "in terms of what is made (a trace) and how it is made (covarying quantities)" (Moore & Thompson, 2015). The term emergent indicates a way of conceiving a graph, either in retrospect or in anticipation, as a trace in the making that arises from images and coordination of covarying quantities (Moore, 2021). This stands in contrast to static graphical shape thinking, in which a graph is conceived as a manipulable shape, such as something that can be translated or rotated (Moore, 2021; Paoletti et al., 2022a). EGST is useful not only for conceptualizing functions (Moore & Thompson,

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2015) but also for supporting the development of other mathematical ideas and for making sense of ideas beyond mathematics (Paoletti et al., 2022b).

Paoletti et al. (2023) extend the idea of EGST by developing a framework that includes three interrelated main components: situational quantitative and covariational reasoning (M.S.), reasoning with graphical representations of covarying quantities (M.R.), and EGST itself (M.E.). According to this framework, for students to engage in M.E., they must engage in reasoning that bridges M.S. and M.R. (Paoletti et al., 2024). Put differently, constructing or interpreting a graph through EGST requires students to reason simultaneously about the covarying quantities situationally and graphically.

Moore & Thompson (2015) hypothesised that EGST has the potential to serve as a productive way of thinking about function transformations when the instructional focus begins with the transformation of the graph itself. They argue that EGST can support students in developing meanings of both what is being transformed and the result of the transformation, namely the graph. This aligns with the findings of Borba & Confrey (1996), who showed that students' engagement in identifying relationships between the values represented by corresponding points across graphs helped them uncover the relationships between their algebraic representations, which constitute the transformation rules. In this way, students can come to understand why those rules work (Yao, 2024).

Learning from dynamic visualizations

Many studies have highlighted the potential of dynamic visualizations to support student learning, particularly for cognitively demanding tasks. The effectiveness of such visualizations depends on at least two factors: their design and their content (Ehrhart & Lindner, 2023). Well-designed dynamic visualizations can help students more easily learn from the information presented (Schnotz & Rasch, 2008). They are also particularly well suited for representing content that is inherently dynamic. Given these affordances, research in mathematics education frequently employs dynamic visualizations to help students make sense of mathematical ideas with dynamic characteristics, including function transformations.

Yao (2024) reports that many dynamic visualizations have been designed to support the teaching and learning of function transformations. Beyond being dynamic, these visualizations are also interactive, providing students with opportunities to explore the transformations themselves. The value of interactivity is echoed in other studies, which suggest that interactive dynamic visualizations can support exploratory learning by enabling students to focus on specific variables of interest, generate hypotheses about relationships among those variables, and evaluate those hypotheses (Bodemer et al., 2004). Such opportunities for exploration are important for students as they learn about function transformations (Anabousy et al., 2014).

Dynamic visualizations, particularly those that present complex information, need to be designed in ways that help students identify relevant information. One way to achieve this is through cueing techniques. Such techniques not only guide students' attention toward essential information but also support them in relating and integrating different elements within the visualization (De Koning et al., 2009). This technique is implicitly suggested by Yao (2024) for designing dynamic visualizations whose goal is to help students discern the critical features of function transformations. To achieve this goal, students need cues that guide their attention to at least one pair of corresponding moving points on the graphs of the original and transformed functions (Yao, 2024). By attending to these points, students can observe how the quantities represented by each pair relate to one another, enabling them to identify the underlying transformation rule and, in turn, to understand why that rule works. In addition, providing a pair of corresponding moving points on the graphs can further support students in conceiving the graphs as representations of covarying quantities, aligning with the EGST framework described earlier.

Current study

This study aims to support students in using EGST to make meaning of horizontal function translation through a digital task. The task was developed in alignment with the theoretical frameworks outlined earlier, namely EGST and research on learning from dynamic visualizations. Guided by this perspective, the study is driven by the following question: How do students use EGST to construct meanings for horizontal function translation when supported by a dynamic visualization?

Methods

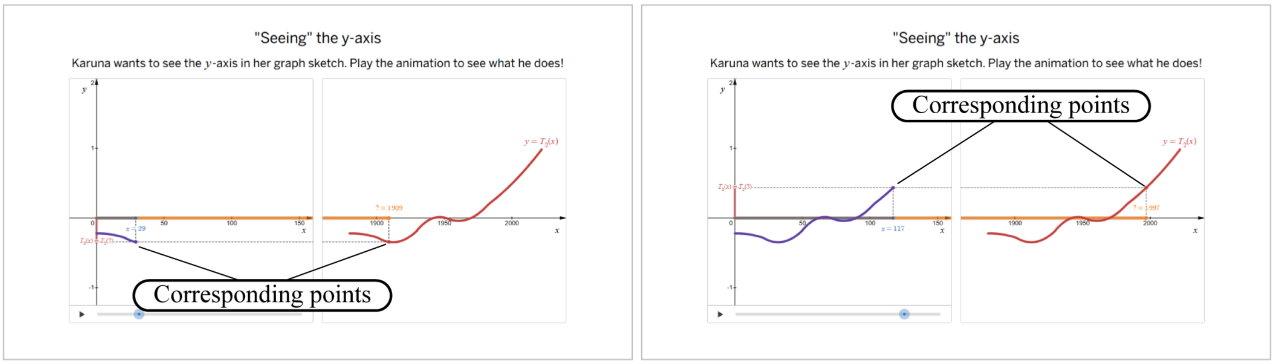
This study draws on a case study approach (Yin, 2018) to examine how two preservice mathematics teachers engaged with a digital task intended to support the development of their meanings for function transformations, with a particular focus on horizontal translation. The case study forms part of a broader research program that employs EGST to strengthen students' understandings of mathematical ideas and their relationships to contexts beyond mathematics.

Participants

The participants were two first-year female preservice mathematics teachers, referred to here by the pseudonyms Anna and Tamara. Before completing the task examined in this study, both had engaged in several EGST-oriented activities, including work on vertical translations of functions. The first author served as the teacher-researcher (TR).

Task design

The task used in this study was the Seeing the y-axis (SY) task. The SY task is part of a larger digital task sequence that connects function modeling and function transformations with a climate-change context (Kristanto & Lavicza, 2025). The goal of the SY task is to guide students in discovering the principle of horizontal function trans-



"Seeing" the y-axis

Karuna wants to see the y-axis in her graph sketch. Play the animation to see what he does!

"Seeing" the y-axis

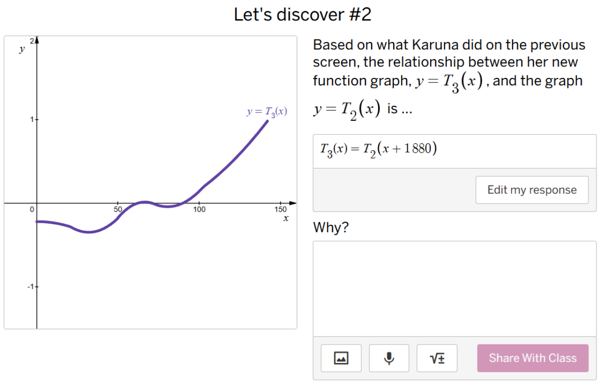
Karuna wants to see the y-axis in her graph sketch. Play the animation to see what he does!

Figure 1. Two moments from the dynamic visualization in the Seeing y-axis task

lation by examining the relationship between an original graph and its horizontally translated graph, together with the corresponding algebraic representations.

To support this goal, the task is framed around a problem that asks students to identify the strategy used by a fictional character, Karuna, to make the y-axis visible. To guide students in identifying this strategy, they are provided with the original graph $y = T_2(x)$, which models temperature anomalies from 1880 to 2022, along with an interactive dynamic visualization that demonstrates how Karuna rescales the x-quantity in T_2 so that the resulting graph, $y = T_3(x)$, begins at the y-axis or $x = 0$. The visualization was designed with three purposes. First, the visualization shows how the graphs T_2 and T_3 are generated by depicting each as the trace of a moving point whose position is constrained by the quantities represented in the graph, which is intended to prompt students to interpret the graphs using EGST. Second, the visualization guides students' attention by highlighting corresponding pairs of points, one on T_2 and one on T_3, that move simultaneously. Third, the visualization is interactive, allowing students to control its flow and enabling exploratory learning. Together, these design considerations aim to support students in using EGST to construct meanings related to horizontal function translation. Figure 1 shows the visualization at two different moments.

By observing how each point on T_3 corresponds to a point on T_2, students are expected to identify the relationship within each pair: the points share the same height, and the difference in their x-coordinates is constant at 1880. From this observation, they are then expected to infer the general relationship $T_3(x) = T_2(x + 1880)$ for every x in the domain of T_3, record this relationship, and provide an explanation in the textboxes provided, as shown in Figure 2.



Let's discover #2

Based on what Karuna did on the previous screen, the relationship between her new function graph, $y = T_3(x)$, and the graph $y = T_2(x)$ is ...

$T_3(x) = T_2(x + 1880)$

Why?

Figure 2. Textboxes for entering the inferred relationship and explanation

Analysis

The data source for this study was a video recording of Anna and Tamara as they worked on the SY task. The recording captured their dialogue, gestures, and interactions with the digital task environment. The video was transcribed to support further analysis. Consistent with a teaching experiment methodology (Steffe & Thompson, 2000), we conducted a retrospective analysis of the data. The first author began by identifying instances that offered insight into Anna's and Tamara's reasoning, and then applied both generative and convergent approaches (Clement, 2000) to develop initial models of the students' meanings. To interpret their mathematical activity, we drew on key components of Paoletti et al.'s (2023) EGST framework, including M.S. (situational quantitative and covariational reasoning), M.R. (reasoning with graphical representations of covarying quantities), and M.E. (emergent graphical shape thinking). The students' interactions with the task were also analyzed to investigate how the design supported the meanings they constructed.

The initial model was then discussed among all authors. The focus of these discussions was to examine whether the model aligned with the students' sense-making by proposing and evaluating alternative explanations of their learning activity (Yin, 2018). This process led to a viable model that accounts for how Anna and Tamara used EGST to develop meanings for function transformations, particularly horizontal translation, through the SY task.

Findings

In this section, we present Anna and Tamara's progression as they worked through the SY task. First, we describe their initial meanings, which led them to reverse the intended relationship between the two functions. Second, we illustrate how their attention to specific input correspondences, supported by instructional scaffolding, helped them refine this relationship. Lastly, we show how these developments enabled them to generalize the relationship between the functions in a way that reflects a coherent meaning of the horizontal translation.

Students' initial meanings and the reversal of the intended relationship

At the beginning of their work on the SY task, Anna and Tamara discussed how to approach the problem. Tamara was the first to propose a solution. Anna disagreed with her proposal, yet their exchange helped surface two important observations about T_3. They noted that the domain of T_3 spans from 0 to 142, and Anna pointed out that the inputs of T_2 and T_3 differ by roughly a thousand units. Their initial discussion is presented in Table 1.

Table 1. Initial discussion of Anna and Tamara

Speaker	Transcript excerpt
Tamara	T_3 is... T_2, minus...
Anna	What?
Tamara	One four two.
Anna	One four two, right, from here to here [*points from the beginning to the end of* T_3*'s domain on the* x*-axis*].
Tamara	Mm-hm. Or minus one...
Anna	No. I don't think so.
Tamara	That's y, $T_3(x)$.
Anna	Mm-hm. So the difference is... this one is from one thousand something... [*points to the left end of the* T_2 *graph*].
Tamara	Yes.
Anna	This one is from zero [*points to the left end of the* T_3 *graph.*]. From zero, right, that's the value.
Tamara	Uh-huh.

There are two central points in Anna and Tamara's discussion in Table 1. First, they shared an understanding of what the task required them to do, namely to express T_3 in terms of T_2. Second, their discussion focused on the difference in the x-coordinates of corresponding points on the graphs of T_2 and T_3. Their search for this difference was grounded in their ability to interpret the graphs. We therefore infer that they were engaged in M.R.

Building on this, Anna searched for a more precise difference between those corresponding x-coordinates. She did so by comparing the left endpoints of the two graphs, i.e. 1880 and 0, and determining their separation, which she found to be 1880. This activity shows that she was able to extract quantitative information from the graphs, particularly from their left endpoints. We therefore infer that she was engaging in quantitative reasoning situationally, that is, M.S. With this in place, she formed an initial conjecture, expressing it tentatively as:

> So T_3 is... T_2 minus... so that the result is zero. 1900. 1880?

We see two possible interpretations of what Anna may have meant. One possibility is that she intended $T_3(x) = T_2(x) - 1880$. Alternatively, she may have been expressing the idea that, to obtain the same T_3 and T_2 values, the x-value for T_3 is equal to the x-value for T_2 minus 1880. In symbolic terms, this can be written as $T_3(x - 1880) = T_2(x)$. We consider the second interpretation more plausible, because it aligns with the reasoning that both students used afterward when testing Anna's initial conjecture.

After formulating her initial conjecture, Anna made use of the interactivity of the dynamic visualization to test it. She moved the playhead back and forth and stopped when the visualization displayed corresponding points with x-coordinates 103.5 on the T_3 graph and 1983.5 on the T_2 graph. She then asked Tamara to compute the difference between these two x-coordinates, saying:

> Try this. This one [*points to 1983.5 on the* T_2 *graph*] minus this one [*points to 103.5 on the* T_3 *graph*]. How much?

Tamara calculated the difference between the two x-coordinates using her calculator, obtained 1880, and reported it to Anna. In response, Anna moved the playhead again, appearing to test her conjecture visually. After a moment, she stated her conjecture to the TR, seeking confirmation of whether it was correct.

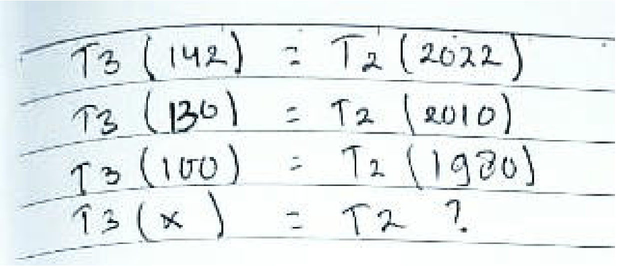
Anna and Tamara's activities in testing Anna's initial conjecture, which centered on determining the difference between the x-coordinates of corresponding points, provide supporting evidence for our interpretation of that conjecture. We interpret her conjecture as expressing the relationship $T_3(x - 1880) = T_2(x)$. In testing it, Anna made use of the interactive features of the visualization, which displayed pairs of corresponding points whose heights were always equal. Anna, assisted by Tamara, then determined the difference between their corresponding x-coordinates and found this difference to be 1880.

Throughout the process of formulating and testing the initial conjecture, there was no evidence that Anna or Tamara referred to the heights or y-coordinates of the corresponding points. This does not imply that they failed to attend to them; rather, we suspect they did not mention them because the dynamic visualization made it clear that the points always shared the same height. Anna's engagement in both M.S. and M.R. while examining the pattern relating corresponding points on T_2 and T_3 was supported by the cueing and interactive features of the dynamic visualization. Her involvement in these two components of EGST may have helped her build reasoning toward M.E.

Anna's engagement in M.S. is particularly interesting for one reason. Her M.S. engagement operated independently of the context underlying the construction of T_2, which represents the relationship between time (in years) and temperature anomaly (in °C). When she extracted information from points on this graph, especially their x-coordinates, she did not explicitly interpret them as times (in years) but instead treated them as real numbers in general. She proceeded similarly when extracting information from points on T_3. Her strategy of treating the x-coordinates of these points as real numbers detached from their contextual meaning, in a way, supported her in identifying the relationship between the corresponding x-coordinates on T_2 and T_3. This approach was efficient for her at that moment to achieve her goal of defining T_3 in terms of T_2. She understood this goal did not require her to make sense of these rules in terms of the quantities to the original context.

Refining the relationship through input correspondences

In the preceding section, we described how Anna and Tamara initially approached the SY task by expressing T_2 in terms of T_3. In response, the TR directed their attention to the hint provided in the visualization, which displayed "$T_3(x) = T_2(?)$" (Figure 1). Using this cue, the TR encouraged them to express T_3 in terms of T_2 rather than the reverse, since T_2 was the given function. The TR then prompted them to use the dynamic visualization to record several pairs of corresponding points from the graphs of T_2 and T_3 and to determine the relationships between the quantities represented by those points. After examining several pairs in the visualization, Anna recorded her observations, as shown in Figure 3.



T3 (142) = T2 (2022)
T3 (130) = T2 (2010)
T3 (100) = T2 (1980)
T3 (x) = T2 ?

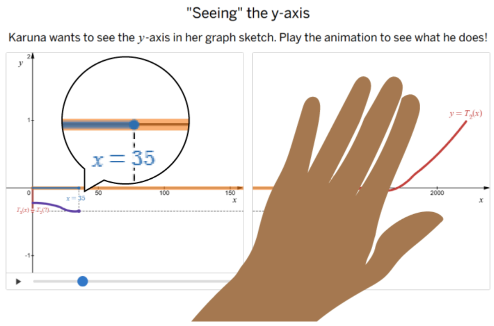
Figure 3. Anna's notes showing the relationship between T_3 and T_2

The notes shown in Figure 3 prompted Anna and Tamara to find a relationship between the inputs of T_3 and those of T_2. After some time, Anna observed that "So every T … T_3 x [*meaning the input of* T_3] increases by 10, then T_2 x [*meaning the input of* T_2] also increases by 10." Based on this observation, she reached a conclusion and stated it aloud to Tamara and the TR, asserting that $T_3(x) = T_2(x)$.

Although Anna's final statement was not a valid conclusion, she was reasoning covariationally as she sought a relationship between the corresponding inputs of T_3 and T_2. In doing so, she identified that a change in one quantity was always accompanied by an equal change in the other. We infer that this perceived sameness in the magnitude of change across the two quantities led her to conclude that $T_3(x) = T_2(x)$. In this moment, her way of reasoning was not yet productive for explicitly expressing the input of T_2 in terms of the input of T_3.

Noticing that Anna and Tamara were still struggling to express the input of T_2 in terms of the input of T_3, TR invited them to revisit the problem. In this discussion, TR adopted a different approach from the one Anna had previously attempted. In this approach, TR encouraged them to determine how each input of T_3 maps to a corresponding input of T_2 by finding the value of the T_2 input for a given value of the T_3 input. This discussion is presented in Table 2.

At the end of the discussion shown in Table 2, Anna and Tamara determined that for any input x of T_3, the corresponding input of T_2 is $x + 1880$. Reaching this conclusion was supported by their meaning that the correspondence between the inputs of T_3 and T_2 could be viewed as an input-output assignment. In this case, the input is the value from T_3 and the output is the associated value from T_2. Using this perspective, they initially identified outputs by directly observing the visualization. Finally, TR prevented them from seeing the output directly by covering the portion of the screen that displayed the T_2 input (Figure 4). This move prompted them to formulate a rule that would allow them to compute the output once the input was known.



"Seeing" the y-axis

Karuna wants to see the y-axis in her graph sketch. Play the animation to see what he does!

Figure 4. Illustration of the TR covering the portion of the screen showing the T2 input

Generalizing the relationship between the functions in the horizontal translation

In the previous section, Anna and Tamara had identified a rule that maps the inputs of T_3 to the corresponding inputs of T_2. Building on this, TR asked them to test the rule using the pairs Anna had previously recorded (Figure 3). They applied the rule to those numerical pairs

Table 2. Discussion revealing how T_3 inputs map to T_2 inputs

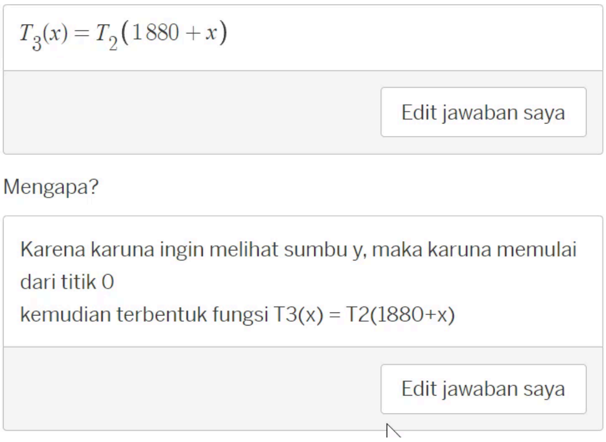
Speaker	Transcript excerpt
TR	If x is zero [*pointing to the x-value of T_3*], here [*pointing to the corresponding x-value of T_2*]?
Anna	Zero [*answered spontaneously*].
Tamara	Zero [*answered spontaneously, as if parroting Anna*].
Anna	1880 [*looks at the T_2 graph and corrects her answer*].
Tamara	1880 [*looks at the T_2 graph also and corrects her answer*].
TR	[*Moves the playhead so the x-value of T_3 is 19*] If here is 19, then?
Tamara	1999, 1899 [*looks at the corresponding x-value of T_2*].
TR	If I cover this [*closing the T_2 graph, then moves the playhead so x of T_3 is 35*], if here is 35, what would it be approximately?
Anna	1880 plus 35... [*muttering*].
Anna	[*Appears to calculate in her notebook, then states the result*] 1915!
TR	How do you do it?
Tamara	Add.
TR	With?
Tamara	With, this one [*points to the graph, but not specifically*]. What I mean is...
Tamara	The 1980 one [*points to the left end of the T_2 graph, meaning she actually refers to 1880*].
Anna	Yes.

and confirmed that it still held. TR then allowed them to use this idea by posing new problems: "For $T_3(10)$, what is it equal to?" and "For $T_3(50)$, what is it equal to in T_2?" They answered both questions correctly without relying on the provided dynamic visualization, but instead used their rule, which showed that they could connect the rule to the relationship between T_3 and T_2. TR then invited Anna and Tamara to generalize their finding so that T_3 could be expressed in terms of T_2. Their discussion is presented in Table 3.

Table 3. Discussion showing how Anna discovered the relationship between T_3 and T_2

Speaker	Transcript excerpt
TR	So, $T_3(x)$ equals?
Tamara	$T_3(x)$...
Anna	1880 plus x [*meaning $T_2(1880 + x)$*].

The discussion in Table 3 shows that Anna, at least, identified the relationship between T_3 and T_2, namely $T_3(x) = T_2(x + 1880)$. This is supported by their responses in the provided text box, which are shown in Figure 5. In addition to stating the relationship between T_3 and T_2, they also provided a justification in the text box. Their Indonesian explanation can be translated as follows: "Since Karuna wanted to see the y-axis, she started from the point 0, which led to the function $T_3(x) = T_2(1880 + x)$."



$T_3(x) = T_2(1880 + x)$

Mengapa?

Karena karuna ingin melihat sumbu y, maka karuna memulai dari titik 0
kemudian terbentuk fungsi T3(x) = T2(1880+x)

Figure 5. Anna and Tamara's written responses

We argue that Anna and Tamara's conclusion about the relationship between T_3 and T_2 emerged from the meanings they had already developed for the two graphs and for the quantities represented in them. As described earlier, the visualization provided in the SY task supported their engagement in the kinds of reasoning that move toward M.E. as they conceived both graphs. This conception, combined with the visualization that directed their attention to corresponding point pairs on T_3 and T_2, helped them identify the relationship between the two functions, even though this relationship was not trivial for them. The visualization allowed them to notice two key patterns in the corresponding points: first, the heights or y-coordinates always matched, and second, there was an invariant pattern in the x-coordinates. By synthesizing these two patterns, they reached the conclusion shown in Figure 5. They then recontextualized this conclusion in terms of the original problem. They reasoned that, to see the y-axis, the new graph needed to begin at $x = 0$, which led them to express the functional relationship as $T_3(x) = T_2(1880 + x)$.

Discussion and conclusion

We have described how students progressed in constructing meanings for horizontal function translation through the case of Anna and Tamara. Their engagement with EGST and with the dynamic visualization played a crucial role in this progression. Through EGST, they conceived both the original graph and its horizontally translated counterpart as the trace of a moving point whose position is constrained by the covarying quantities represented in each graph. Supported by the cueing features of the dynamic visualization, they were then able to identify the relationship between the quantities represented by corresponding pairs of points on the original and translated graphs. They subsequently used this relationship to express the translated function in terms of the original, $T_3(x) = T_2(1880 + x)$. Considering the trajectory of their reasoning, we argue that they held a meaning for this equation by conceiving, graphically, the height of T_3 at any value x as matching the height of T_2 at the corresponding value $1880 + x$. Thus, their discovery of the horizontal translation rule was supported by what Paoletti et al. (2023) describe as reasoning with graphical representations. Oehrtman et al. (2008) describe this kind of meaning as a powerful way of making sense of horizontal function translation and as reinforcing a process view of functions.

The interactivity of the dynamic visualization supported Anna and Tamara in exploring and identifying the relationship between a function and its horizontal translation. The interactive features enabled them to formulate an initial conjecture based on their observations of the visualization. In addition, they used these features to generate alternative cases to test whether those cases were consistent with their initial conjecture. The interactive features were also leveraged by the TR to provide scaffolding that helped Anna and Tamara revise their initial conjecture until they reached an intended conclusion. This aligns with findings from previous studies suggesting that interactive dynamic visualizations can offer opportunities for exploratory learning (Bodemer et al., 2004; Wichmann & Timpe, 2015).

Our study shows that Anna and Tamara's construction of meaning about horizontal function translation is not straightforward. This is evident in their initial conjecture, which described the known function in terms of its translated version, the opposite of what we intended. We suspect this happened because horizontal translation and horizontal transformations, more generally, carry an inherent conceptual complexity. In such transformations, students must express the transformed function in relation to the original, but they must do so by describing the original function's input in terms of the transformed function's input. In other words, the direction of the relationship between the functions is reversed relative to the relationship between their respective inputs, a conceptual shift that can pose challenges for students. This possible explanation extends those offered in prior studies that have identified sources of students' difficulties in learning horizontal function transformations (Yim & Lee, 2022; Zazkis et al., 2003).

Generalizing the pattern between the corresponding inputs of the original function and its translated version was also not straightforward for Anna and Tamara. They first reasoned covariationally to understand how these input values related, eventually noticing that the changes in the two inputs had the same magnitude. They were then supported in repeatedly determining the input of the original function, given the input of the translated function. This process led them to identify what Smith (2008) refers to as a correspondence relationship. We therefore argue that opportunities to discern how one value depends on another, provided repeatedly across varied input pairs, supported their discovery of this correspondence relationship. This is consistent with findings from previous studies (Chimoni et al., 2023; Ureña et al., 2024).

Our study carries several implications. We suggest that students' engagement in EGST while interpreting the graphs of an original function and its horizontal translation supports the development of productive meanings for this transformation. This stands in contrast to approaches that introduce horizontal translation by treating the graph as a shape to be shifted (i.e., through static graphical shape thinking), a method that can contribute to the counterintuitive nature of the transformation. With respect to dynamic visualizations, we propose that interactivity and cueing on corresponding points between the original and translated graphs are crucial for helping students discern the critical relationships involved in horizontal translation. In addition, the interactivity of such visualizations could be further enhanced so that students can more readily identify the corresponding relationship between the x-coordinates of corresponding points. For example, providing an option for students to hide or display the x-coordinates of the original function could support this process.

Before closing this article, we acknowledge several limitations of the study. First, the meanings for horizontal function transformation that we have described are based on the reasoning of two students; other students may construct different meanings. Future research could investigate how students from other populations develop meanings for this transformation when supported by EGST. Second, the meanings constructed by Anna and Tamara in this study arose from their interpretations of the graphs presented in the dynamic visualization. It would be valuable to examine how such meanings might emerge not from interpreting given graphs, but from constructing the graph of the translated function on the basis of the original graph.

Our study extends ongoing lines of inquiry into how students can be supported in making sense of horizontal translations of functions, how EGST can foster students' understanding of graphs and related mathematical ideas, and how students learn with dynamic visualizations. From this study, we are persuaded that EGST can support students in developing productive meanings related to horizontal function translation when instruction begins with the graph and is then followed by the algebraic representation. Future research can continue this line of inquiry and potentially broaden it to investigate whether EGST is similarly productive for supporting students' meaning making for other function transformations or function operations more generally.

References

Anabousy, A., Daher, W., Baya'a, N., & Abu-Naja, M. (2014). Conceiving Function Transformations in Different Representations: Middle School Students Working with Technology. *International Electronic Journal of Mathematics Education*, *9*(2), 97–112. https://doi.org/10.29333/iejme/284

Baker, B., Hemenway, C., & Trigueros, M. (2001). On transformations of basic functions. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra: Proceedings of the 12th ICMI Study Conference: The Future of the Teaching and Learning of Algebra*.

Bodemer, D., Ploetzner, R., Feuerlein, I., & Spada, H. (2004). The active integration of information during learning with dynamic and interactive visualisations. *Learning and Instruction*, *14*(3), 325–341. https://doi.org/10.1016/j.learninstruc.2004.06.006

Borba, M. C., & Confrey, J. (1996). A student's construction of transformations of functions in a multiple representational environment. *Educational Studies in Mathematics*, *31*(3), 319–337. https://doi.org/10.1007/BF00376325

Božić, R., Takači, Đ., & Stankov, G. (2021). Influence of dynamic software environment on students' achievement of learning functions with parameters. *Interactive Learning Environments*, *29*(4), 655–669. https://doi.org/10.1080/10494820.2019.1602842

Chimoni, M., Pitta-Pantazi, D., & Christou, C. (2023). Unfolding algebraic thinking from a cognitive perspective. *Educational Studies in Mathematics*, *114*(1), 89–108. https://doi.org/10.1007/s10649-023-10218-z

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 547–589). Routledge.

De Koning, B. B., Tabbers, H. K., Rikers, R. M. J. P., & Paas, F. (2009). Towards a Framework for Attention Cueing in Instructional Animations: Guidelines for Research and Design. *Educational Psychology Review*, *21*(2), 113–140. https://doi.org/10.1007/s10648-009-9098-7

Ehrhart, T., & Lindner, M. A. (2023). Computer-based multimedia testing: Effects of static and animated representational pictures and text modality. *Contemporary Educational Psychology*, *73*, 102151. https://doi.org/10.1016/j.cedpsych.2023.102151

Kristanto, Y. D., & Lavicza, Z. (2025). Down-to-earth mathematics: transforming curriculum and teaching on function transformations. In *Proceedings of the Fourteenth Congress of the European Society for Research in Mathematics Education (CERME14)*. Free University of Bozen-Bolzano; ERME. https://hal.science/hal-05294249

Lage, A. E., & Gaisman, M. T. (2006). An Analysis of Students Ideas About Transformations of Functions. In *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 23–30). PME-NA.

Moore, K. C. (2021). Graphical Shape Thinking and Transfer. In C. Hohensee & J. Lobato (Eds.), *Transfer of Learning* (pp. 145–171). Springer International Publishing. https://doi.org/10.1007/978-3-030-65632-4_7

Moore, K. C., & Thompson, P. W. (2015). Shape thinking and students' graphing activity. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education* (pp. 782–789). West Virginia University.

Oehrtman, M., Carlson, M., & Thompson, P. (2008). Foundational Reasoning Abilities that Promote Coherence in Students' Function Understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (1st ed., pp. 27–42). The Mathematical Association of America. https://doi.org/10.5948/UPO9780883859759.004

Paoletti, T., Gantt, A. L., & Corven, J. (2023). A Local Instruction Theory for Emergent Graphical Shape Thinking: A Middle School Case Study. *Journal for Research in Mathematics Education*, *54*(3), 202–224. https://doi.org/10.5951/jresematheduc-2021-0066

Paoletti, T., Hardison, H. L., & Lee, H. Y. (2022a). Students' static and emergent graphical shape thinking in spatial and quantitative coordinate systems. *For the Learning of Mathematics*, *42*(2), 48–50. https://flm-journal.org/index.php?do=show&lang=en&vol=42&num=2

Paoletti, T., Lee, H. Y., Rahman, Z., Vishnubhotla, M., & Basu, D. (2022b). Comparing graphical representations in mathematics, science, and engineering textbooks and practitioner journals. *International Journal of Mathematical Education in Science and Technology*, *53*(7), 1815–1834. https://doi.org/10.1080/0020739X.2020.1847336

Paoletti, T., Stevens, I. E., Acharya, S., Margolis, C., Olshefke-Clark, A., & Gantt, A. L. (2024). Exploring and promoting a student's covariational reasoning and developing graphing meanings. *The Journal of Mathematical Behavior*, *74*, 101156. https://doi.org/10.1016/j.jmathb.2024.101156

Ross, J. A., Bruce, C. D., & Sibbald, T. M. (2011). Sequencing computer-assisted learning of transformations of trigonometric functions. *Teaching Mathematics and Its Applications*, *30*(3), 120–137. https://doi.org/10.1093/teamat/hrr009

Schnotz, W., & Rasch, T. (2008). Functions of animations in comprehension and learning. In R. Lowe & W. Schnotz (Eds.), *Learning with animation: Research implications for design* (pp. 92–113). Cambridge University Press.

Steffe, L. P., & Thompson, P. W. (2000). Teaching Experiment Methodology: Underlying Principles and Essential Elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 267–307). Erlbaum.

Thompson, P. W. (2015). Researching Mathematical Meanings for Teaching. In L. D. English & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education* (0th ed., pp. 447–473). Routledge. https://doi.org/10.4324/9780203448946-27

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421–456). National Council of Teachers of Mathematics.

Ureña, J., Ramírez, R., Molina, M., & Cañadas, M. C. (2024). Generalization: strategies and representations used by sixth to eighth graders in a functional context. *Mathematics Education Research Journal*, *36*(3), 519–545. https://doi.org/10.1007/s13394-023-00458-w

Wichmann, A., & Timpe, S. (2015). Can Dynamic Visualizations with Variable Control Enhance the Acquisition of Intuitive Knowledge?. *Journal of Science Education and Technology*, *24*(5), 709–720. https://doi.org/10.1007/s10956-015-9554-8

Yao, X. (2024). Opportunity to Learn Function Transformations in the Open-Access GeoGebra Resources. *Digital Experiences in Mathematics Education*, *10*(1), 108–131. https://doi.org/10.1007/s40751-024-00136-x

Yim, J., & Lee, H. S. (2022). Students' Use of Prior Conceptions of Symbols in Finding an Equation for a Horizontal Translation of the Graph of a Function. *International Journal of Science and Mathematics Education*, *20*(8), 1699–1717. https://doi.org/10.1007/s10763-021-10230-w

Yin, R. K. (2018). *Case study research and applications: design and methods* (Sixth edition). SAGE.

Zazkis, R., Liljedahl, P., & Gadowsky, K. (2003). Conceptions of function translation: obstacles, intuitions, and rerouting. *The Journal of Mathematical Behavior*, *22*(4), 435–448. https://doi.org/10.1016/j.jmathb.2003.09.003