Down-to-earth mathematics: transforming curriculum and teaching on function transformations

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The present study proposes and evaluates a digital task sequence aimed at transforming the curriculum and teaching of function transformations by promoting productive meanings of these transformations and integrating mathematics with other domains, specifically climate science. The findings of this study show how students' envisioning of a graph in terms of what is made (a trace) and how it is made (covarying quantities) supports their construction of productive meanings of function transformations. Furthermore, this study also highlights challenges experienced by students during the learning process, which can be used to improve future instruction.

Keywords: boundary crossing, DNR, emergent graphical shape thinking, function transformations.

Introduction

Function transformations are often taught by first introducing a parent function and treating other functions as transformations of this parent shape. While this approach can enhance students' fluency in applying function transformations, researchers (Moore & Thompson, 2015) argue that it may not fully support students' understanding of what they are transforming—specifically, representations of covarying quantities. This study proposes and evaluates a digital task sequence aimed at transforming the curriculum and teaching of function transformations by promoting productive meanings of these transformations and integrating mathematics with other domains, specifically climate science. The task sequence incorporates emergent graphical shape thinking (EGST) as a boundary object, providing students with opportunities to navigate between the domains of mathematics and climate science within a DNR-based instructional context.

In the following sections, we first describe the theoretical frameworks that informed the design of the digital task sequence. We then detail the structure and rationale behind the task sequence, illustrating its role in supporting students' understanding of function transformations as transformations of covarying quantities.

Positioning EGST within DNR and Boundary Crossing Frameworks

Our design integrates two frameworks: DNR-based instruction in mathematics (Harel, 2008) and boundary crossing (Akkerman & Bakker, 2011), along with its associated learning mechanism, particularly reflection. We draw on DNR's principles of duality, necessity, and repeated reasoning to inform the design of our digital task sequence. Drawing on the principle of duality, the task sequence focuses on both developing meanings of function transformations and using EGST as a way of thinking. Drawing on the principle of necessity, we designed the task sequence to include problems that ignite students' intellectual needs. Drawing on the principle of repeated reasoning, we structured the task sequence to provide opportunities for students to use EGST repeatedly throughout the tasks.

Building on the boundary crossing framework, we design the task sequence to support students in reflecting, through taking and making perspectives on both mathematics and climate science, in order to develop meanings of function transformations. We hypothesize that EGST can serve as a boundary object, facilitating students' perspective-taking and perspective-making across the domains of mathematics and climate science. This aligns with previous studies showing that graphical reasoning is an important tool for understanding both mathematics and science concepts (Paoletti et al., 2024; Stephens, 2024).

From both frameworks, EGST plays an important role: as a way of thinking to construct meanings of function transformations and as a boundary object bridging the domains of mathematics and climate science. We refer EGST as "envisioning a graph in terms of what is made (a trace) and how it is made (covarying quantities)" (Moore & Thompson, 2015). Paoletti et al. (2023) argue that for students to engage in EGST, they must engage in both situational and graphical reasoning. We integrate these forms of thinking and reasoning throughout our digital task sequence, which is described in the following section.

Digital task sequence

Our digital task sequence is titled "Down-To-Earth Mathematics: How Hot Is Our Earth?". The phrase "down-to-earth mathematics" was chosen for two reasons: first, to make mathematics more relatable and realistic for students, and second, to use the context of the Earth to raise awareness about the real and urgent challenges related to global temperature trends. We designed the task sequence in Desmos, and it is publicly accessible at https://s.id/down2earth-math.

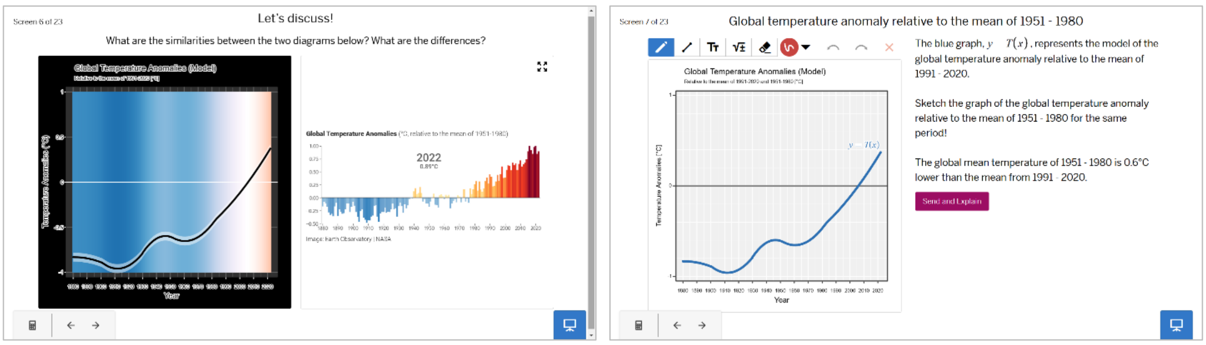


Figure 1: Sample screens from Task VT

The task sequence guides students in using perspectives from climate science to interpret global temperature anomalies over time, and then making mathematical perspectives to construct meanings of function transformations from contextualized problem situations. This understanding of function transformations is subsequently applied to taking mathematical perspectives on function transformations, allowing students to make connections in climate science and predict future global temperature anomalies. This approach highlights the opportunities for students to reflect on both mathematics and climate science through the processes of perspective-taking and perspective-making. Building on this approach, the blueprint of the task sequence is outlined in Table 1.

Table 1: Blueprint for the digital task sequence

Task	Description
MT	**Modeling global temperature anomalies.** Students watch a video animation showing global temperature anomalies (relative to the 1991–2020 average) from 1880 to 2022, featuring both a heatmap and warming stripes. They record what they notice and wonder about the animation. Then, they sketch a graph to model the global temperature anomalies over time.
VT	**Building meanings of vertical translation.** Students compare and contrast a model based on the 1991–2020 average with another graphical representation using a different baseline (1951–1980 average) for global temperature anomalies. Using the information that the 1951–1980 average is 0.6°C lower than the 1991–2020 average (see Figure 1), they then sketch a graph modeling global temperature anomalies relative to the 1951–1980 average. Finally, students relate the resulting model to the previous one to develop an understanding of vertical translation, both visually and algebraically.
HT	**Building meanings of horizontal translation.** Students are presented with a problem of "seeing the y-axis," requiring them to modify the previous model so that it starts from $x = 0$ (the domain of the previous model is from 1980 to 2022). They watch an animation illustrating the construction of the new model alongside the previous one (see Figure 2). Finally, they connect the new model to the previous one to develop a visual and algebraic understanding of horizontal translation.
VS	**Building meanings of vertical shrinking/stretching.** Students sketch a graph of a new model where global temperature anomalies are measured in °F, based on the previous model in °C. They then connect the new model to the previous one, developing a visual and algebraic understanding of vertical stretching/shrinking.
HS	**Building meanings of horizontal shrinking/stretching.** Students watch a video animation demonstrating a strategy for adjusting the x-axis scale to match the y-axis scale. The video illustrates the construction of a new graph alongside the previous model. Students then connect the new model to the previous one, developing a visual and algebraic understanding of horizontal stretching and shrinking.
PT	**Predicting future global temperature anomalies.** Students predict the trend of future global temperature anomalies by extending the graph of the previous model. They then observe the intersection of the graph with $x = 2.25$. Using their understanding of function transformations, they identify the year corresponding to $x = 2.25$.

The design of the task sequence presented in Table 1 follows DNR's instructional principles of duality, necessity, and repeated reasoning (Harel, 2008). First, it addresses both the meanings of function transformations and the use of EGST as a way of thinking. EGST is incorporated in two ways: (a) by asking students to sketch a graph based on a given situation (MT, VT, and VS, see Figure 1), or (b) by having students watch an animation of the graph's construction (HT and HS, see Figure 2). Second, the task sequence creates intellectual needs for students to develop their understanding of function transformations: (a) changing the baseline of temperature anomalies for vertical translations; (b) "seeing the y-axis" problem for horizontal translations; (c) converting temperature units for vertical stretching/shrinking; and (d) adjusting the x-axis scale for horizontal stretching/shrinking. Finally, throughout the task sequence, EGST is repeatedly incorporated to build students' fluency.

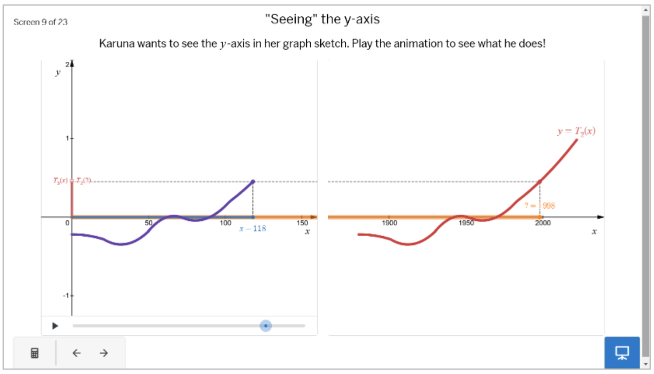


Figure 2: A sample screen from Task HT

Research questions

While the DNR and boundary crossing frameworks inform the design of the task sequence, the EGST framework plays a crucial role in guiding the design. Therefore, this study narrows its focus to investigating the opportunities the task sequence provides in promoting EGST to support students in constructing meanings of function transformations. The research questions guiding this evaluation are: How, and to what extent, do students develop their meanings of function transformations through the use of EGST?

Methods

This study is part of a larger design research project aimed at transforming curriculum and instruction related to change and relationships. Specifically, it presents a lesson case showcasing the implementation of a digital task sequence on function transformations. The participants were 22 first-year undergraduate students enrolled in an algebra and trigonometry course at a private university in Yogyakarta, Indonesia.

Prior to these lessons, the students had already engaged with EGST in interpreting and constructing graphs, as reported elsewhere (Kristanto et al., 2024). The lessons were conducted in two sessions in early October 2023. The first author, who also served as the teacher-researcher, was the regular course instructor. At the beginning of the first session, the teacher explained temperature anomalies, including their definition and how they are calculated, as this was an essential prerequisite for the tasks. Students worked in groups of two or three, resulting in nine groups. In the first session, they completed Tasks MT, VT, and HT; in the second session, they worked on Tasks VS, HS, and PT. During the sessions, the teacher facilitated productive mathematics discussions by allowing students to work in groups while monitoring their progress, purposefully selecting and sequencing groups' work for presentation, and connecting students' approaches and the underlying mathematics (Stein et al., 2008).

Students' work on the digital platform Desmos was recorded. Additionally, the teacher's Desmos dashboard was recorded to provide supplementary data on students' progress. To analyze how students developed their meanings of function transformations, we coded their works and video recordings of the teacher's dashboard using the EGST framework (Paoletti et al., 2023). The codes comprised situational quantitative and covariational reasoning (MS), reasoning with graphical representations of covarying quantities (MR), and emergent graphical shape thinking (ME). To assess the extent of their understanding, we focused on evaluating students' answers to Tasks VT, HT, VS, and HS, as these tasks align directly with the research questions guiding this study. Using the rubric presented in Table 2, we analyzed their responses to these tasks, where students were asked to (1) represent their new model algebraically and (2) explain their reasoning.

Table 2: Rubric

Mark allocated	Description for algebraic representation	Description for reasoning
0	No answer or incorrect solution.	No answer or incorrect solution.
1	Correct solution.	The answer demonstrates a conceptual understanding of the problem, as reflected in the mathematical approach used. However, no evidence links the reasoning to the problem's context exists.
2	–	The answer demonstrates a complete understanding of the problem, is accurate, and includes clear reasoning that is well-connected to the context of the problem.

Results

We begin by qualitatively examining how students develop their meanings of function transformations. Next, we quantitatively assess the extent to which they have developed these meanings.

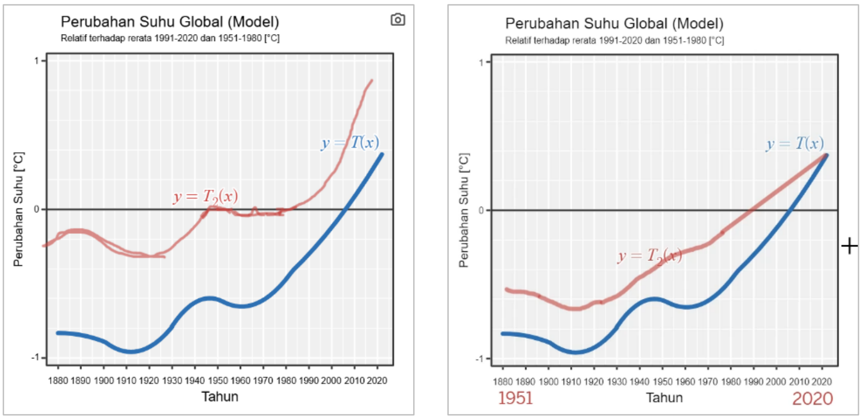
How students develop their meanings of function transformations

Nearly all groups accurately sketched continuous graphs when tasked with modeling annual global temperature anomalies in Task MT. Only one group, Group 9, plotted discrete points but followed the correct trend. This suggests that all groups demonstrated meanings of MS and MR, while one group did not engage in ME as they did not account for all possible points.

In Task VT, six groups adopted a strategy to shift the graph upwards by 0.6 units when the baseline was changed from 1951–1980 to 1991–2020 in their initial sketch. The reasoning from these groups was similar. One group explained their approach as follows:

> Due to the difference in the average global temperature anomalies, the period from 1950 to 1980 differs by 0.6 degrees from the average global temperature changes between 1991 and 2020. This causes the entire temperature comparison graph to shift upwards, indicating that temperatures are getting hotter (Group 7, translated from Indonesian).

Group 7's reasoning indicates engagement with MS, MR, and ME. They appeared to conceive each point of the previous model (with the 1991–2020 baseline) as representing a co-ordination between years and global temperature anomalies, treating each point as a multiplicative object. Using this meaning, they sketched a new model (with the 1951–1980 baseline) by shifting all points of the previous model upward by 0.6, see Figure 3(a). This demonstrates that they developed a meaning of what the y-coordinates in each model represent—specifically, the global temperature anomalies based on the 1991–2020 and 1951–1980 baselines, respectively.



(a) (b)

Figure 3: Sketches from two groups for Task VT

Group 1 employed a different strategy by initially plotting the graph from 1951 to 1980 and then extending it to cover 1880 to 2022, as shown in Figure 3(b). This group likely had an incomplete meaning of the baseline in calculating temperature anomalies. Two other groups drew a new graph by shifting the old one downwards. Their rationale for this strategy was the same: the average temperature from 1951–1980 was 0.6°C lower than from 1991–2020. This suggests that the groups lacked a comprehensive meaning of ME, particularly regarding the concept of temperature anomaly and its relationship to the baseline.

In Task HT, all groups arrived at the correct answer, $T_3(x) = T_2(x + 1880)$. One group used the explanation of a "shifted reference point" as follows:

> This is because Karuna assumes that the year 1880 represents zero, so the intervals for subsequent years start from 1880 as the reference point (Group 5, translated from Indonesian).

For Task VS, while all groups successfully sketched the transformation graph, only a few accurately explained their strategies. For example, one group explained it as follows:

> If we assume that 0°C corresponds to 0°F, then if Celsius increases to 1°C, Fahrenheit will rise by 9/5 or 1.8 units. This means the graph shifts approximately 1.8 units downward (if Celsius is negative) or upward (if Celsius is positive) (Group 9, translated from Indonesian).

In Task HS, eight groups wrote $T_5(x) = T_4(71x)$, but only three groups provided accurate reasoning. One group explained it as follows:

The graph of $T_5(x)$ is derived from the graph of $T_4(x)$ by compressing the input (x) by a factor of 71 (Group 4, translated from Indonesian).

In summary, students' meanings in MS, MR, and ME supported their development of function transformation meanings. Students who understood function transformations as transformations of covarying quantities envisioned each point on the graph as a multiplicative object, which helped them relate it to the new graph. For example, in Task HS, they observed that each y-coordinate in the new model corresponded to the y-coordinate at 71 times the x-coordinate in the previous model.

To what extent did students develop meanings of function transformations?

Figure 4 shows the distribution of group scores across four tasks (VT, HT, VS, and HS). The students performed best in Task VT, where most groups scored 2, indicating they made correct algebraic work with sound mathematical reasoning but did not connect their reasoning to the context of changing the temperature anomaly baseline. In contrast, Task VS saw lower scores, with most groups receiving a score of 1. While these groups demonstrated correct algebraic work, they did not provide reasoning to explain their approach.

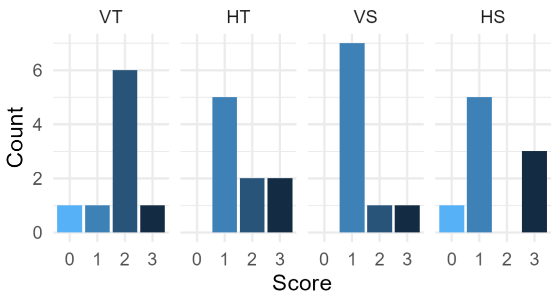


Figure 4: Frequency of group scores across tasks, with scores ranging from 0 to 3

Discussion and conclusion

The findings of this study highlight the potential of the proposed digital task sequence in supporting students in developing productive meanings of function transformations. The qualitative data analysis revealed that EGST supports students in understanding the relationship between a function and the result of its transformation, both algebraically and visually. The results of the quantitative data analysis further reinforce this finding. The findings of this study reveal that some students still demonstrated an incomplete understanding of the baseline used in temperature anomaly calculations. These misconceptions hindered their ability to fully develop mathematical concepts, particularly concerning vertical shifts in functions. This insight highlights areas for improvement in refining the digital task sequence for future iterations. Additionally, an interesting case emerged with Group 1. Their sketch in Task MT provided evidence of their meanings of MS, MR, and ME. However, their incomplete understanding of a critical quantity, i.e. temperature anomaly, prevented them from succeeding in Task VT, which required a proper understanding of this concept.

In refining our task sequence, we plan to incorporate a strategy from cognitive load theory to minimize the extraneous cognitive load associated with understanding the concept of temperature anomaly (Chen et al., 2023). Specifically, we will redesign the sequence to include an initial phase where students learn how to calculate temperature anomalies using a given baseline, followed by opportunities to practice these calculations independently. We hypothesize that this approach will help reduce their cognitive load when working on Task VT.

While this study focuses on how students construct meanings of function transformations through EGST, it does not extensively analyze how students navigate disciplinary boundaries between mathematics and climate science. Future research could explore how integrating real-world contexts, such as climate change, fosters or hinders students' meanings of function transformations.

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