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EXPLORING STUDENTS' UNDERSTANDING OF LINEAR AND QUADRATIC RELATIONSHIPS IN A PROJECTILE MOTION CONTEXT

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Abstract
Previous research has shown that students often struggle to develop an understanding of linear and quadratic relationships. Covariational reasoning has been identified as a way to support this development. This study aims to investigate how covariational reasoning supports students in developing understandings of linear and quadratic relationships within a projectile motion context. A teaching experiment was conducted with two middle school students who engaged in a digital task exploring the relationship between height and time. The analysis characterizes how the students' covariational reasoning evolved as they interpreted the changing quantities in the task. The findings suggest that prompts encouraging students to compare linear and quadratic relationships can foster more sophisticated forms of covariational reasoning. The discussion highlights how specific features of the task design, including the affordances of technology, the emphasis on conceiving graphs as representations of covarying quantities, and the use of non-canonical graphing tasks, can support covariational reasoning.

Keywords: covariational reasoning, linear relationship, projectile motion, quadratic relationship

Introduction

Linear and quadratic relationships are essential for interpreting real-world phenomena (Abrams, 2023; Stewart et al., 2016), yet many studies have shown that students continue to struggle in understanding these relationships (Van Dooren et al., 2008). For instance, students often struggle to connect the algebraic forms of linear and quadratic relationships with their graphical representations and the real-world situations they model (Leinhardt et al., 1990; Wilkie, 2019; Wilkie & Ayalon, 2018). These challenges suggest the need for instructional approaches that help students develop deeper, more connected understandings of how quantities change together, such as through covariational reasoning.

Covariational reasoning, or reasoning about how two quantities change in tandem, is fundamental for students' understanding of mathematical relationships (Thompson & Carlson, 2017). Using this reasoning allows them to interpret functions dynamically, bridging algebraic expressions, graphs, and contextual situations. Prior studies have shown that promoting covariational reasoning can help students develop an understanding of the rate of change (Tallman et al., 2024; Yu, 2024), concepts that underlie both linear and quadratic relationships. Consequently, fostering covariational reasoning may be a productive way to address students' challenges in understanding linear and quadratic relationships.

Therefore, this study employed a teaching experiment to characterize how students' covariational reasoning supports their development of understanding of linear and quadratic relationships. We drew on the covariational reasoning framework (Thompson & Carlson, 2017) to analyze how students reasoned about varying quantities as they engaged with tasks involving these relationships. This analysis allowed us to trace how their reasoning evolved and supported conceptual development. The study offers empirical insights into the role of covariational reasoning in fostering students' understanding of fundamental functional relationships in mathematics. The following sections review relevant literature and theoretical perspectives before describing the methods and presenting the findings.

Literature Review

Previous research has highlighted the potential of covariational reasoning to support students in developing meaningful understandings of mathematical relationships. One productive approach for fostering such reasoning is through the use of real-world phenomena. For example, Sandoval Jiménez and Sierra (2025) designed a modeling task based on a Newtonian problem in which students were asked to calculate temperatures beyond the range measurable by a thermometer. They found that engaging in this modeling task promoted students' covariational reasoning. Similarly, Yu (2025) and Altindis (2025) incorporated motion-based contexts into their tasks to foster covariational reasoning. Such contexts draw on students' intuitive understandings of physical phenomena, allowing them to focus on coordinating quantities that are familiar to them (Yu, 2025). Such real-world contexts can also generate an intellectual need for students to make sense of mathematical ideas through covariational reasoning (Harel, 2013; Paoletti, Moore, et al., 2024).

However, students often encounter difficulties when modeling real-world phenomena, particularly those involving motion contexts. A common challenge is that students tend to interpret graphs representing quantities in such phenomena using a spatial coordinate system, even when this interpretation is inappropriate (Paoletti, Hardison, et al., 2022). In other words, students often treat the graph as a literal depiction of the phenomenon, interpreting it as a picture or map of the physical situation rather than as a representation of how quantities covary. One promising approach to address this issue is to encourage students to conceive of a graph both in terms of what is produced (a trace) and how it is produced (via covarying quantities) (Moore & Thompson, 2015). Another is to employ non-canonical graphing tasks, for example, those that represent time on the vertical axis and height on the horizontal axis. These unconventional representations can challenge students' assumptions and engage them in novel reasoning processes, including quantitative and covariational reasoning, even when they are familiar with conventional graph forms (Moore et al., 2014). By disrupting familiar visual associations, such tasks can promote more flexible and conceptual understandings of functional relationships.

Previous studies have also leveraged the affordances of technology to promote students' covariational reasoning in modeling dynamic situations. For example, Johnson (2020) used technology to display a video animation of a moving car alongside a sketching panel that allowed students to construct graphs representing the quantities involved in the situation. This design provided opportunities for students to engage in covariational reasoning. Pittalis et al. (2025) employed similar technological affordances to facilitate students in graphically representing motion scenarios involving covarying quantities. However, their task design further utilized the affordance of directly linking the graph to the animation, allowing students to observe how changes in one representation corresponded to changes in the other. Such uses of technology illustrate its potential not only to visualize dynamic relationships but also to create integrated environments where students can coordinate multiple representations (Ainsworth, 1999), thereby deepening their understanding of how quantities change together.

Taken together, these studies suggest that engaging students with real-world phenomena, encouraging them to conceive of graphs as representations of covarying quantities, and introducing non-canonical graphing tasks can provide meaningful contexts for promoting covariational reasoning. Moreover, the affordances of technology can further enhance these learning experiences by enabling students to dynamically link representations and observe quantitative changes in real time.

Theoretical framework

Covariational reasoning refers to the ability to mentally envision two or more quantities changing at the same time, maintaining a continuous awareness of their values and how those values vary together (Saldanha & Thompson, 1998). Thompson and Carlson (2017) describe covariational reasoning as progressing through several developmental levels, summarized in Table 1, ranging from a lack of coordination between varying quantities to a sophisticated understanding of two variables varying smoothly and continuously.

Table 1. Covariational Reasoning Framework (Thompson & Carlson, 2017)

Level	Description
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.
Pre-coordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.

Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, non-multiplicative link between the overall changes in two quantities' values.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.

The framework of covariational reasoning presented in Table 1 does not explicitly include the rate of change. According to Thompson and Carlson (2017), conceptualizing rate of change requires covariational reasoning but also involves more conceptualizations, such as ratio and quotient. Accordingly, attending to how students quantify a rate of change can offer valuable insights into their covariational reasoning and inform how instruction might better support students in reasoning at more advanced levels (Yu, 2024).

Current study

The present study aims to explore how covariational reasoning supports students in developing an understanding of linear and quadratic relationships in a projectile motion context. Therefore, this study addresses the following research question: How does covariational reasoning support middle school students in developing meanings for linear and quadratic relationships in a projectile motion context?

Methods

This study employed a teaching experiment (Steffe & Thompson, 2000) to characterize two students' activity as they engaged with the *Height-Time Relationship Task*. The task was part of a larger project aimed at promoting emergent graphical shape thinking, in which students interpret and construct graphs representing covarying quantities (Kristanto et al., 2025; Kristanto & Lavicza, 2025). The teaching experiment was conducted by the first author, who served as the teacher-researcher (TR) and facilitated the students' engagement with the task. In this study, we examined how covariational reasoning supported students in developing meanings for linear and quadratic relationships. The following sections describe the participants, the task, and the analytical approach.

Participants

The participants were two female ninth-grade students from a public middle school in Yogyakarta, Indonesia. TR was not the students' regular classroom teacher, which helped create an exploratory learning environment distinct from their usual classroom setting. The students, referred to by the pseudonyms Fania and Bianca, were selected based on two criteria. First, they represented diverse academic abilities. Second, both demonstrated strong verbal communication skills, which were important for generating rich verbal data during the teaching experiment. The selection of participants was assisted by the students' regular mathematics teacher, who identified candidates meeting these criteria. The participating school was chosen for its familiarity with using technology, such as laptops or tablets, in classroom instruction.

The Height-Time Relationship Task

The Height-Time Relationship Task was part of a larger digital task sequence titled "How Accurate Is Your Aim?"[1] which was designed to provide a game-based learning environment that supports students' engagement in emergent graphical shape thinking as they interpret and construct graphs related to projectile motion. The task sequence was developed in Desmos Classroom (now Amplify Classroom).

The design of the task sequence was informed by suggestions from previous research. First, it employed a real-world context, projectile motion, that could serve as a meaningful reference for students as they engaged in mathematical reasoning (Czocher et al., 2022; Sokolowski, 2021b). Second, it included non-canonical tasks that presented quantities in unfamiliar ways. For example, in typical projectile motion problems, time is represented on the x-axis and height on the *y*-axis. In this task sequence, the axes were reversed, with height represented on the *x*-axis and time on the y-axis. This unconventional representation was intended to engage students in new reasoning processes, such as quantitative and covariational reasoning (Moore et al., 2014). Third, the sequence also aimed to promote students' engagement in emergent graphical shape thinking as they interpreted and constructed graphs (Moore & Thompson, 2015; Paoletti et al., 2023). Finally, the design utilized the affordances of technology to synchronize the animation of the physical phenomenon with its graphical representation (Ainsworth, 1999; Johnson et al., 2020; Pittalis et al., 2025). For instance, when the ball moved upward, the line segment representing its height became longer, and when the ball descended, the line segment became shorter. These features were intended to help students connect the motion they observed with its quantitative representation.

[1] The complete digital task sequence can be accessed online at https://classroom.amplify.com/activity/69061b958b3deaa21d53e9ef

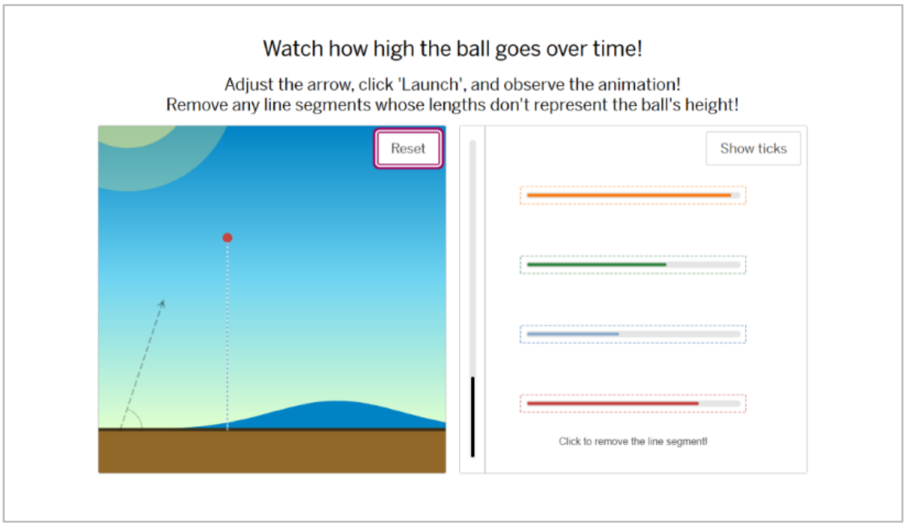

Figure 1. Screen Showing the Launch Activity Where Students Select a Line Segment Representing the Ball's Height Over Time

Within this sequence, the Height-Time Relationship Task consisted of three interactive screens that guided students to explore relationships between height and time during the motion of a launched ball. On the first screen (Figure 1), students could adjust the length and direction of an arrow to launch a ball. As the ball moved through its trajectory, they were asked to observe its height over time and to choose one of four line segments that best represented the ball's changing height. To support decision-making, students could eliminate the options one by one until only a single line segment remained.

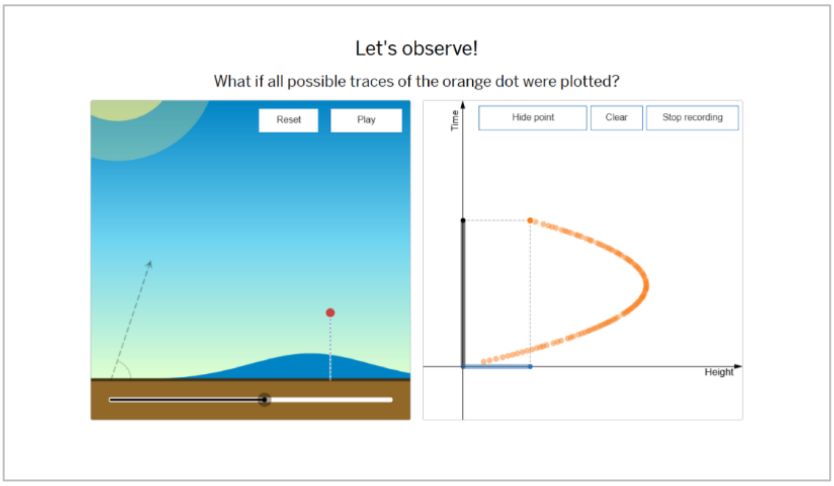

Figure 2. Screen Showing the Dynamic Relationship Between Height and Time Using Moving Line Segments

On the second screen (Figure 2), students again launched the ball and observed how its height changed over time. In this activity, the two quantities, height and time, were represented as the lengths of line segments along the *x*- and *y*-axes. Students could represent the relationship between these quantities as a moving point whose motion was constrained by the two line segments on the axes. They also had the option to trace the path of the moving point, allowing them to visualize the trajectory it followed.

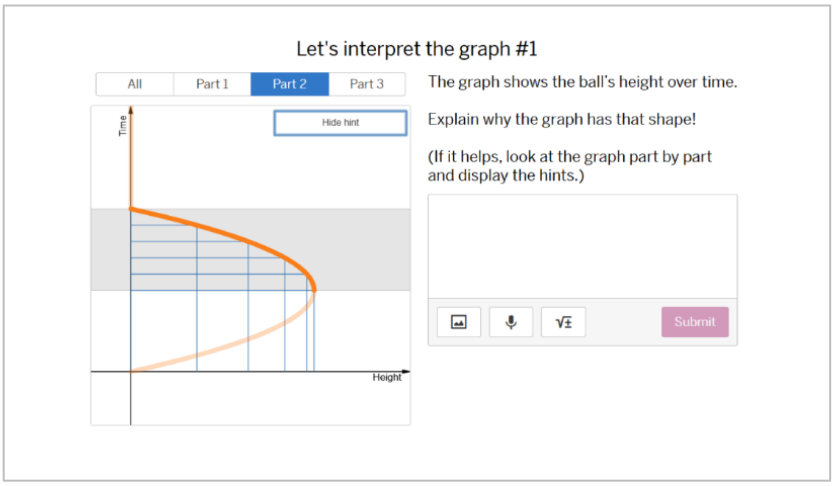

Figure 3. Graph Interpretation Screen Where Students Analyze the Ball's Height-Time Graph

The third screen (Figure 3) displayed a graph showing the height of the ball as a function of time and prompted students to interpret the shape of the graph. Students could analyze the graph in sections and were offered hints consisting of several equally spaced points on the y-axis, along with their corresponding points on the *x*-axis. These hints supported students in connecting graphical features to the quantitative relationships represented.

Analysis

During data collection, the two students sat side by side and worked collaboratively on a single laptop to complete the task. TR sat in front of them to facilitate the teaching experiment. While the students worked, their laptop screen was recorded using screen recording software, and their facial expressions were captured using the built-in webcam. A separate camera positioned behind them recorded their gestures and audio. All recordings were synchronized into a single composite video to support detailed analysis. Based on this video, we selectively transcribed the dialogue between TR and the students, along with the students' gestures, focusing on their activity on the third screen of the task. In particular, we focused on coverbal gestures, or gestures produced in coordination with speech, as these often convey meanings that complement verbal expressions and reflect students' emerging reasoning processes (Kendon, 2000; Kong et al., 2017; Paneth et al., 2024).

In analyzing the data, we first identified instances in the students' activity that provided evidence of their covariational reasoning as they engaged with the third screen. This screen was chosen because it was where the students developed their understanding of linear and quadratic relationships through graph interpretation. Second, we coded these instances using Thompson and Carlson's (2017) covariational reasoning framework. To strengthen the reliability of our analysis, we maintained a clear evidentiary trail that linked the data and our interpretations directly to the research questions. Finally, we connected the students' covariational reasoning with how they constructed understandings for linear and quadratic relationships.

Findings and Discussion

This section presents the activity of Fania and Bianca as they engaged with the third screen of the Height-Time Relationship Task (Figure 3). The findings are reported chronologically to trace the shift in their reasoning. First, we describe how the students interpreted what the graph was composed of without yet connecting it to the covarying quantities it represented. Second, we examine their emerging understandings of the two covarying quantities represented in the graph. Finally, we present how their covariational reasoning developed as they compared the graph with a linear function graph.

Interpreting the Structure of the Graph

When Fania and Bianca opened the third screen, TR asked them to describe what the displayed graph represented. Responding to this prompt, Fania said:

> "Because it's like a collection of recorded data. Uh, what is it... possibilities. A trace, like a record of all the possible points where the points could be."

We interpreted Fania's understanding as being influenced by her experience on the previous screen (Figure 2). On that screen, she had observed how a graph representing the ball's height over time was formed as a trace of a moving point whose motion was determined by the quantities it represented, namely height and time. Based on her statement, we infer that Fania conceived of the graph as traces of a point whose position was determined by the two varying quantities, the ball's height and time. This interpretation of a graph as a trace of dynamic quantities is consistent with the perspectives suggested by Moore and Thompson (2015) and Paoletti et al. (2023).

TR then asked what shape the graph had. Fania responded:

> "Like a half circle, curved, kind of oval [makes an oval gesture with her right thumb and index finger], and then it's straight [uses her left hand to make a vertical line gesture]."

Bianca agreed with Fania's interpretation. Their description of the graph's shape was notable for two reasons. First, neither of them referred to it as a parabola, which suggests that the graph appeared novel to them. Second, their use of the term half oval to describe its form indicates that they drew on a familiar visual shape, likely a horizontally oriented half oval, to describe what they observed.

Constructing Meanings for Covarying Quantities

TR then invited Fania and Bianca to focus more closely on interpreting the shape of the graph. Specifically, TR asked why the graph appeared as a half oval at first, and was then followed by a vertical line. This question initiated the discussion presented in Table 2.

Table 2. Excerpt of Students' Discussion When Interpreting the Shape of the Graph

Speaker	Transcript
Fania	Mmm, maybe it's also affected by height. Because at the beginning, it starts from below [places right hand low], then it's thrown [makes a gesture of throwing a ball upward]. Since it's thrown, the height on the graph moves to the right. At a certain point, mmm, it comes back down because it's pulled by gravity [uses hand to trace the motion of a ball going up and down like a parabola]. So it comes back, and the graph moves to the upper left [uses right hand to make a diagonal gesture upward to the left]. That means the ball is going down [mimics a falling motion with her hand]. Then, when it reaches the ground, there's no more change in the ball's height, but time keeps moving, so the graph becomes a straight line.
TR	Bianca, anything to add?
Bianca	[Shakes her head] No, it's the same.

Based on the discussion in Table 2, Fania was able to describe how the quantities of time and the ball's height changed together. In doing so, she coordinated the physical situation, the motion of the ball over time, with the graph that represented it. For instance, she associated the upward-left segment of the graph with the moment when the ball was descending. This interpretation provides strong evidence that she was able to make meaningful sense of the graph and engage in covariational reasoning. Rather than construing the coordinate system spatially, which is a common tendency among students who interpret graphs as literal depictions of physical trajectories (Clement, 1989; Hadjidemetriou & Williams, 2002; Paoletti, Lee, et al., 2022), she conceptualized it quantitatively, with height represented on the x-axis and time on the y-axis. Within this understanding, she appropriately interpreted the upward-left segment of the graph as representing a decrease in height as time progressed, corresponding to the ball's downward motion. However, there was no evidence that she constructed a multiplicative object that explicitly paired time and height values. Therefore, we infer that she was engaging in covariational reasoning, particularly at the stage of gross coordination of values.

Shifts in Covariational Reasoning through Comparison with a Linear Graph

To promote more advanced covariational reasoning, TR invited Fania and Bianca to compare the graph on the third screen with a graph that the TR drew on the whiteboard. On the whiteboard, TR sketched a graph that followed the same overall pattern previously described by Fania: first extending upward to the right, then upward to the left, and finally continuing with a vertical line upward. However, unlike the digital graph, all parts of TR's graph followed linear patterns (Figure 4).

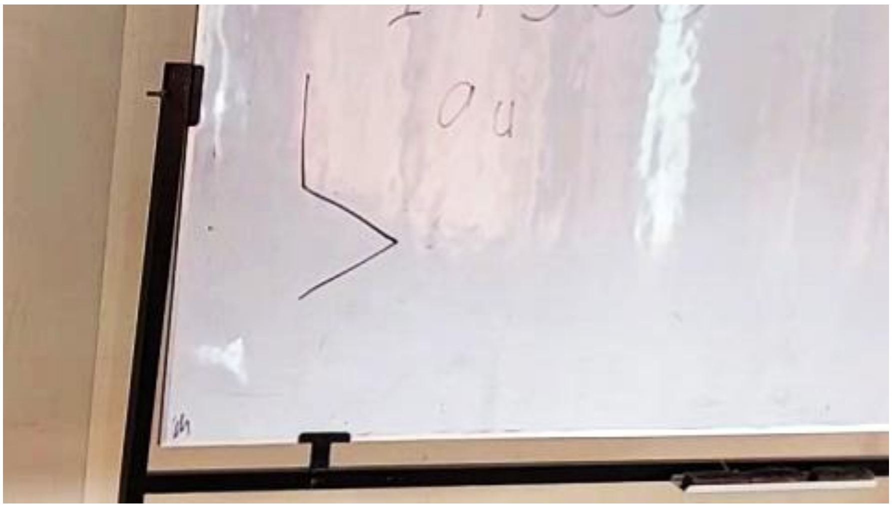

Figure 4. The Graph Drawn by TR on the Whiteboard

TR then asked why the graph on the third screen appeared curved rather than composed of straight segments, as in Figure 4. This question prompted Fania and Bianca to interpret TR's graph and led to the discussion presented in Table 3.

Table 3. Discussion Comparing the Curved and Linear Graphs

Speaker	Transcript
Fania	If it's diagonal, if it's straight like that [pointing to the TR's graph on the whiteboard], that means the speed is constant.
Bianca	Yes.
Fania	Every second it--
Bianca	Every second it--
Fania	Same.
Bianca	Same.

Based on Table 3, Fania and Bianca identified the key characteristic of the graph drawn by TR, namely that it represented a linear function. They interpreted that such a graph depicts two quantities whose average change is constant. In this reasoning, they were not only engaged in covariational reasoning but also made use of the ratio between the change in one quantity, the ball's height, and the change in the other quantity, time. Specifically, they used "every second" as the unit of time. Therefore, we infer that they were engaged in covariational reasoning at least at the stage of chunky continuous covariation.

After that, TR asked Fania and Bianca to interpret the graph on the third screen again, but this time part by part, using the hints provided. The following excerpt shows Fania's interpretation of the first part of the graph, which represents the ball's height over time as it moves upward.

> "Oh, that means within the same unit of time, the height isn't necessarily the same. If the graph were a straight diagonal line like that [pointing to the graph on the whiteboard], then within the same time interval, the height would always be the same, so the differences between intervals would be equal. But because this graph curves, that means for the same unit of time, the height varies. So, in the first second,

> for example, it's still far apart because it's just been thrown [making a throwing gesture from bottom to top], so the height increases quickly. In the second second, it gets slower [gesturing smaller intervals with her right thumb and index finger]. So the graph becomes more compact, see [gesturing again to show smaller intervals with her left hand]. In the third second, it's even smaller because it slows down again [repeating the gesture with her right hand]. In the fourth second, smaller again [same gesture]. Until finally it stops at the top [pointing upward with her right hand], then goes down again [mimicking the downward motion of the ball]."

TR then invited Bianca to add to or restate Fania's explanation in her own words. Bianca responded as follows:

> "So, if the diagonal is a bit curved, that means each second has a different height, not the same. If it's a straight line like this [moving her right hand upward diagonally to form a straight line], that means every second the height is the same, like it goes straight up [repeating the same gesture]. And, what's it called, when the distance between them gets smaller [gesturing smaller intervals with her right thumb and index finger], that means it's about to go down."

Based on Fania and Bianca's explanations, they contrasted the graph on the third screen with the linear graph drawn on the whiteboard. Specifically, they compared how one quantity changed while the other varied at a constant rate. They interpreted that the ball's height changed by progressively smaller amounts as time increased uniformly. Fania referred to "the first second," "the second second," "the third second," and "the fourth second" to indicate that she was considering equal units of time, even though the intervals shown by the hints on the third screen did not necessarily represent one second. Thus, we infer that both students engaged in covariational reasoning and that their reasoning showed a shift toward a more sophisticated coordination of the two varying quantities. Whereas earlier they demonstrated reasoning at the stage of gross coordination of values, they now exhibited reasoning characteristic of the chunky continuous covariation stage.

Fania and Bianca applied similar reasoning when interpreting the second part of the graph, which represented the ball's descent. Their explanations reflected covariational reasoning at the stage of chunky continuous covariation. TR then invited one of them to interpret the final segment of the graph. Bianca responded as follows:

> "This means that the ball is no longer moving and has stopped, but time keeps passing." [She moved her right hand upward to illustrate the increase in time along the *y*-axis.]

Bianca's explanation indicates her understanding of the vertical line segment on the graph. She interpreted the vertical segment as representing a constant height while time continued to progress. In doing so, she no longer referred to discrete time units to explain the ball's motion but instead expressed that the height remained unchanged as time flowed continuously. We therefore infer that she engaged in covariational reasoning at the smooth continuous covariation stage, demonstrating a more refined coordination of continuously varying quantities.

Discussion

We have shown how Fania and Bianca's covariational reasoning developed as they interpreted the relationship between height and time in the context of projectile motion. Our findings indicate that prompts encouraging students to compare linear and quadratic relationships can foster more sophisticated forms of covariational reasoning. Moreover, such reasoning supported the students in constructing meaning for a linear relationship as one in which the change in one quantity remains constant when the change in the other quantity is constant. In other words, they understood a linear relationship as one characterized by a constant rate of change.

Building on this understanding, the students developed meanings for the relationship between height and time in projectile motion: that the change in height becomes smaller as the ball rises and larger as it falls. Although this understanding alone does not fully characterize the quadratic relationship that models projectile motion, it may serve as a productive entry point, or in Harel's (2013) terms, an intellectual need, for exploring additional characteristics of quadratic relationships. Thus, our study contributes to the growing body of research that employs covariational reasoning as a lens for supporting students in making sense of different types of mathematical relationships (Hohensee et al., 2024; Paoletti & Vishnubhotla, 2022; Sokolowski, 2021a; Wilkie, 2020).

In our findings, Fania and Bianca engaged in covariational reasoning both when imagining the motion of the ball (whether rising, falling, or stationary) and when interpreting the graph. We highlight several important points regarding this. First, the opportunity to observe the physical situation helped them construct the involved quantities and the relationships between them. In our study, this was supported by the technological affordances that allowed the situation to be displayed dynamically while simultaneously showing the corresponding quantitative representations. This aligns with prior studies demonstrating the role of technology in promoting covariational reasoning (Johnson et al., 2020; Martínez-Sierra & Villadiego Franco, 2025; Pittalis et al., 2025). Second, the students' understanding of the graph as a representation of covarying quantities, consistent with the perspective suggested by Moore and Thompson (2015), supported their engagement in covariational reasoning. This finding resonates with previous research highlighting the importance of helping students view graphs as representations of covarying quantities rather than as static shapes (Moore, 2021; Paoletti, Hardison, et al., 2022; Paoletti, Stevens, et al., 2024). Finally, our study also shows that the use of non-canonical graphing tasks can be effectively implemented with middle school students, providing them with opportunities to engage in novel and productive forms of mathematical reasoning (Moore et al., 2014, 2019). In addition, the use of such tasks in our study provided strong evidence for assessing whether students engaged in covariational reasoning when interpreting graphs. The non-canonical design enabled us to distinguish whether students were reasoning within a spatial coordinate system or constructing a quantitative coordinate system as they analyzed the projectile motion context.

Future research could extend this work by exploring students' covariational reasoning in analyzing other representations of projectile motion, such as horizontal distance versus time. In doing so, students' understanding of height-time and horizontal distance-time relationships may support their reasoning about parametric representations of projectile motion. Another promising direction is to further investigate how students develop an understanding of quadratic relationships as involving a constant rate of change of the rate of change.

This study has several limitations. The analysis focused only on the students' activity on the third screen, so it does not capture how their reasoning may have been influenced by earlier screens of the same task. However, this focus still provided meaningful insights into how covariational reasoning supports students in developing an understanding of linear and quadratic relationships. In addition, the study involved only two students, which limits the generalizability of the findings, as other students may develop different ways of understanding these relationships.

Conclusion

This study has illustrated a developmental shift in how middle school students engage in covariational reasoning when interpreting relationships between height and time in a projectile motion context. By comparing linear and quadratic relationships, students developed an understanding of constant and varying rates of change, which supported their construction of meaningful connections between quantities. The findings highlight the potential of technology-enhanced, non-canonical graphing tasks for fostering students' covariational reasoning in dynamic situations.

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